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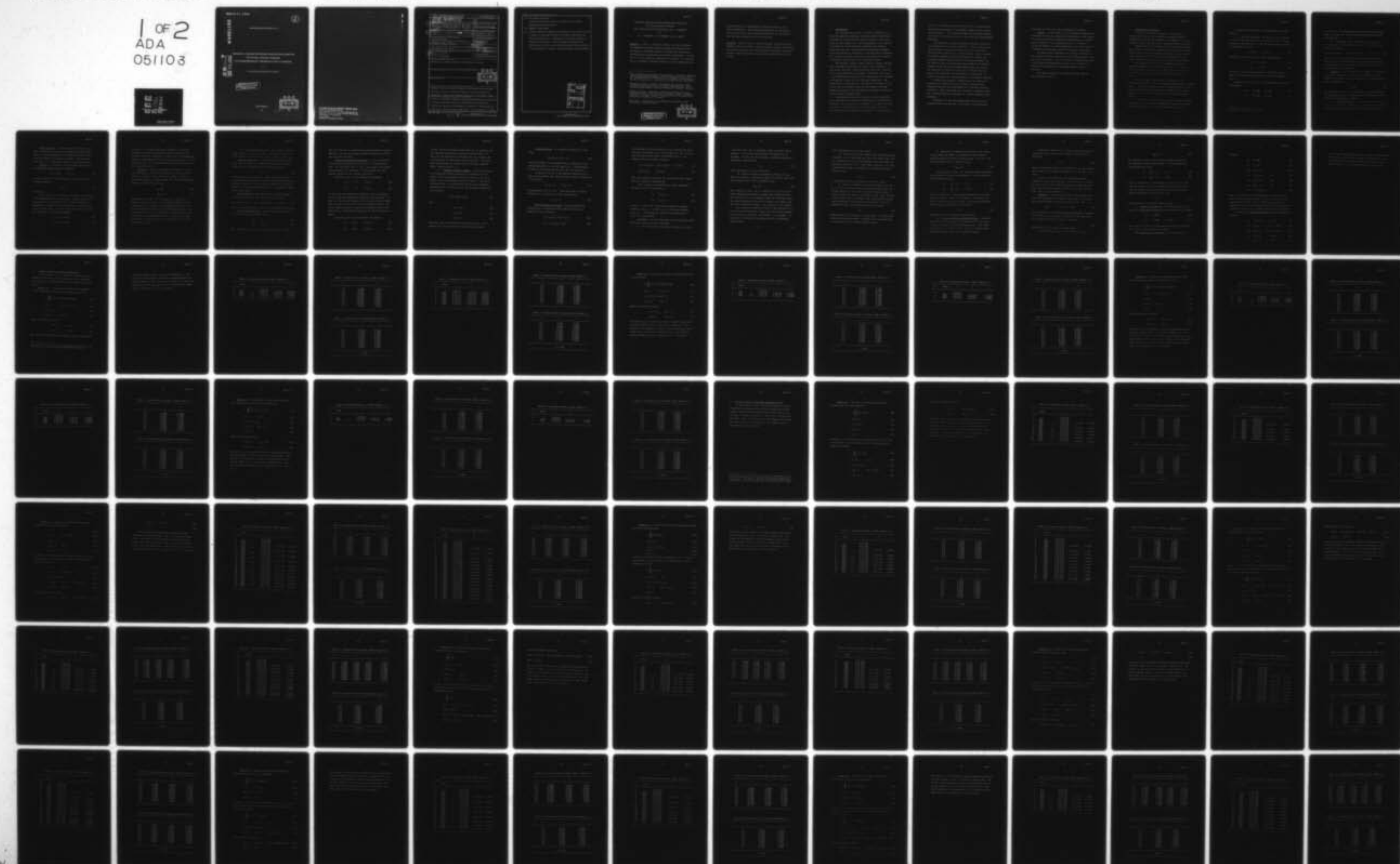

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AERO-ASTRONAUTICS REPORT NO. 127

# SEQUENTIAL CONJUGATE GRADIENT-RESTORATION ALGORITHM FOR OPTIMAL CONTROL PROBLEMS WITH NONDIFFERENTIAL CONSTRAINTS, PART 2, EXAMPLES

by

J.R. CLOUTIER, B.P. MOHANTY, and A. MIELE







RICE UNIVERSITY

1977

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Optimal control, gradient methods, conjugate-gradient methods, numerical methods, computing methods, gradient-restoration algorithms, sequential gradient-restoration algorithms, (OVER)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)


In ~~Ref. 1, Cloutier, Mohanty, and Miele developed~~ the first volume the sequential conjugate gradient-restoration algorithm was developed for minimizing a functional subject to differential constraint, nondifferential constraints, and terminal constraints. In this report, (OVER)


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sequential conjugate gradient-restoration algorithms, nondifferential constraints.

20. ABSTRACT (Continued)

sixteen numerical examples are presented, four pertaining to a quadratic functional subject to linear constraints and twelve pertaining to a nonquadratic functional subject to nonlinear constraints. These examples demonstrate the feasibility as well as the rapid convergence characteristics of the sequential conjugate gradient-restoration algorithm.

# Sequential Conjugate Gradient-Restoration Algorithm for Optimal Control Problems with Nondifferential Constraints, Part 2, Examples[1]

by

J.R. CLOUTIER[2], B.P. MOHANTY[3], and A. MIELE[4]

Abstract. In Ref. 1, Cloutier, Mohanty, and Miele developed the sequential conjugate gradient-restoration algorithm for minimizing a functional subject to differential constraints, nondifferential constraints, and terminal constraints. In this report, sixteen numerical examples are presented, four pertaining to a quadratic functional subject to linear constraints and

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[2] Graduate Student, Department of Mathematical Sciences, Rice University, Houston, Texas. Presently, Mathematician, Naval Surface Weapons Center, Dahlgren Laboratory, Dahlgren, Virginia.

[3] Graduate Student, Department of Mechanical Engineering and Materials Science, Rice University, Houston, Texas. Presently, Assistant Professor, School of Systems Science, Arkansas Tech University, Russelville, Arkansas.

[4] Professor of Astronautics and Mathematical Sciences, Rice University, Houston, Texas.

twelve pertaining to a nonquadratic functional subject to nonlinear constraints. These examples demonstrate the feasibility as well as the rapid convergence characteristics of the sequential conjugate gradient-restoration algorithm.

Key Words. Optimal control, gradient methods, conjugate-gradient methods, numerical methods, computing methods, gradient-restoration algorithms, sequential gradient-restoration algorithms, sequential conjugate gradient-restoration algorithms, nondifferential constraints.

## 1. Introduction

In a previous report (Ref. 1), Cloutier, Mohanty, and Miele developed the sequential conjugate gradient-restoration algorithm for solving optimal control problems in which the state x(t), the control u(t), and the parameter $\pi$ are required to satisfy not only differential constrains and terminal constraints but also nondifferential constraints everywhere along the interval of integration. The algorithm is composed of a sequence of cycles, each cycle consisting of two phases, a conjugate gradient phase and a restoration phase.

The conjugate gradient phase involves a single iteration and is designed to decrease the value of the functional. In this single iteration, nominal functions x(t), u(t), $\pi$ satisfying all the constraints are assumed. Variations $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$ are determined so that the first variation of the functional is minimized, subject to the linearized constraints. The minimization is performed over the class of variations of the control and the parameter which are equidistant from some constant multiple of the corresponding variations of the previous conjugate gradient phase.

The restoration phase involves one or more iterations and is designed to restore the constraints to a predetermined accuracy. In each restorative iteration, the constraint error is reduced. More precisely, nominal functions x(t), u(t), $\pi$ not satisfying the constraints are assumed. Variations $\Delta x(t)$,

$\Delta u(t)$, $\Delta\pi$ are determined so that the norm of the variations of the control and the parameter is minimized, subject to the linearized constraints. The restoration phase is terminated whenever the norm of the constraint error is less than some predetermined tolerance.

The sequential conjugate gradient-restoration algorithm is characterized by two main properties. First, at the end of each conjugate gradient-restoration cycle, the trajectory satisfies the constraints to a given accuracy; thus, a sequence of feasible suboptimal solutions is produced. Second, the conjugate gradient stepsize and the restoration stepsize can be chosen so that the restoration phase preserves the descent property of the conjugate gradient phase; thus, the value of the functional at the end of any cycle is smaller than the value of the functional at the beginning of that cycle. Of course, restarting the algorithm might be occasionally necessary.

To facilitate numerical integrations, the interval of integration is normalized to unit length. Variable-time terminal conditions are transformed into fixed-time terminal conditions. Then, the actual time at which the terminal boundary is reached becomes a component of a vector parameter being optimized.

Convergence is attained whenever both the norm of the constraint error and the norm of the error in the optimality

conditions are less than some predetermined tolerances.

Outline. In this report, sixteen numerical examples are developed in order to illustrate the feasiblity of the sequential conjugate gradient-restoration algorithm and its convergence properties. Four examples pertain to a quadratic functional subject to linear constraints, and twelve pertain to a nonquadratic functional subject to nonlinear constraints.

The experimental conditions are outlined in Section 2. The numerical examples pertaining to the linear-quadratic case are given in Section 3, and the numerical examples pertaining to the nonlinear-nonquadratic case are given in Section 4. Finally, the discussion and the conclusions are given in Section 5.

The symbols employed in this paper are the same as those employed in Ref. 1.

## 2. Experimental Conditions

In order to evaluate the theory, sixteen numerical examples were solved. The sequential conjugate gradient-restoration algorithm was programmed in FORTRAN IV, and the numerical results were obtained in double-precision arithmetic.

Computations were performed at Rice University, Houston, Texas, using an IBM 370/155 computer, and at the Naval Surface Weapons Center, Dahlgren Laboratory, Dahlgren, Virginia, using a CDC-6700 computer. The computations presented here are those obtained at Rice University on the IBM 370/155 computer.

Each example was run with the sequential conjugate gradient-restoration algorithm (SCGRA) of Ref. 1. For comparison purposes, each example was also run with the sequential ordinary gradient-restoration algorithm (SOGRA) of Ref. 2.

For each example, the interval of integration was divided into 100 steps. The differential equations were integrated using Hamming's modified predictor-corrector method (Ref. 3), with a special Runge-Kutta starting procedure. The definite integrals I, J, P, Q were computed using a modified Simpson's rule.

For both the restoration phase and the conjugate gradient phase, the linear, two-point boundary-value problem was solved employing the method of particular solutions (Refs. 4-7).

A detailed description of the experimental conditions follows.

2.1. Decision Variables. The major decision variables controlling the algorithm are the constraint error P and the optimality condition error Q. The following tolerance levels are chosen for P and Q:[5]

$$\varepsilon_1 = E\text{-}08, \qquad \varepsilon_2 = E\text{-}04 \ . \tag{1}$$

Depending on the value of P, two cases are possible:

$$\text{(i)} \qquad P > E\text{-}08, \tag{2}$$

$$\text{(ii)} \qquad P \leq E\text{-}08 \ . \tag{3}$$

In Case (i), the algorithm executes a restoration phase. In Case (ii), the algorithm computes the optimality condition error Q.

Depending on the value of Q, two subcases of Case (ii) are possible:

$$\text{(iii)} \qquad P \leq E\text{-}08, \qquad Q > E\text{-}04, \tag{4}$$

$$\text{(iv)} \qquad P \leq E\text{-}08, \qquad Q \leq E\text{-}04. \tag{5}$$

---

[5]The symbol E$\pm$ab stands for $10^{\pm ab}$.

In Case (iii), the algorithm executes the conjugate gradient phase. In Case (iv), the algorithm stops; convergence has been achieved.

2.2. Iterations. Each iteration of the conjugate gradient phase or the restoration phase is described by the following relations:

$$\tilde{x}(t) = x(t) + \alpha A(t), \qquad \tilde{u}(t) = u(t) + \alpha B(t), \qquad \tilde{\pi} = \pi + \alpha C, \tag{6}$$

which tie the nominal functions and the varied functions. Therefore, each iteration includes two distinct operations: the determination of the basic functions $A(t)$, $B(t)$, $C$ and the determination of the stepsize $\alpha$.

Safeguard. *Let N denote the total number of iterations* (the sum of the number of restorative iterations and the number of conjugate gradient iterations). The following upper bound is imposed on $N$:

$$N \leq 100. \tag{7}$$

Any violation of Ineq. (7) is indicative of extreme slowness of convergence; hence, it constitutes a nonconvergence condition for the algorithm as a whole.

2.3. Restoration Phase. The restoration phase includes one or more restorative iterations. A restorative iteration is started whenever the constraint error $P$ satisfies Ineq. (2).

Basic Functions. In each restorative iteration, the basic functions $A(t)$, $B(t)$, $C$ are determined by solving the linear, two-point boundary-value problem (43)-(49) of Ref. 1 using the method of particular solutions. This requires executing $n + p + 1$ independent sweeps of the above system.

Stepsize. The stepsize $\alpha$ must be determined so that the following inequalities are satisfied:

$$\tilde{P}(\alpha) < \tilde{P}(0), \qquad \tilde{\tau}(\alpha) \geq 0. \tag{8}$$

For this purpose, a bisection process, starting from the reference stepsize

$$\alpha_0 = 1, \tag{9}$$

is employed.

In the course of a restorative iteration, the reduction of the constraint error is guaranteed. However, there is no guarantee that the constraint error is reduced below the threshold (3) characterizing the beginning of the next conjugate gradient phase. In other words, after Ineqs. (8) have been satisfied, two cases are possible:

$$\text{(i)} \quad \tilde{P}(\alpha) > \text{E-08}, \tag{10}$$

$$\text{(ii)} \quad \tilde{P}(\alpha) \leq \text{E-08}. \tag{11}$$

In Case (i), a further restorative iteration is initiated employing as nominal functions the varied functions of the previous restorative iteration. In Case (ii), the restoration phase is terminated, and the next conjugate gradient phase is started. Therefore, Ineq. (11) constitutes the convergence condition of the restoration phase.

Safeguards. Within each restoration phase, let $N_r$ denote the number of restorative iterations. Within each restorative iteration, let $N_{br}$ denote the number of stepsize bisections required to satisfy Ineqs. (8). The following upper bounds are imposed on $N_r$ and $N_{br}$:

$$N_r \leq 10 , \tag{12}$$

$$N_{br} \leq 10 . \tag{13}$$

Any violation of Ineq. (12) is indicative of failure to produce a feasible solution in a reasonable number of iterations; hence, it constitutes a nonconvergence condition for both the restoration phase and the algorithm as a whole. Any violation of Ineq. (13) is indicative of extreme smallness of the restorative displacements; hence, it constitutes a nonconvergence condition for both the restoration phase and the algorithm as a whole.

2.4. Conjugate Gradient Phase. The conjugate gradient phase involves a single iteration. This single iteration is started whenever the constraint error P satisfies Ineq. (3).

Auxiliary Functions. In each conjugate gradient iteration, the first step is to compute the auxiliary functions $A_*(t)$, $B_*(t)$, $C_*$ corresponding to the fictitious value

$$\gamma_* = 0 \tag{14}$$

of the directional coefficient. These auxiliary functions are determined by solving the linear, two-point boundary-value problem (144)-(150) of Ref. 1 using the method of particular solutions. This requires executing $n + p + 1$ independent sweeps of the above system.

Basic Functions. With the auxiliary functions known, the basic functions are determined with the relations

$$A(t) = A_*(t) + \gamma\hat{A}(t), \qquad B(t) = B_*(t) + \gamma\hat{B}(t), \qquad C = C_* + \gamma\hat{C}, \tag{15}$$

where $\gamma$ denotes the actual value of the directional coefficient.

Directional Coefficient. The directional coefficient $\gamma$ is set at one of the following levels:

$$\text{(i)} \qquad \gamma = 0, \tag{16}$$

$$\text{(ii)} \qquad \gamma = Q/\hat{Q}, \tag{17}$$

where (16) holds for the first conjugate gradient iteration

and (17) holds for any subsequent conjugate gradient iteration. Therefore, the first conjugate gradient iteration is an ordinary gradient iteration.

Slope of the Augmented Functional. Prior to accepting the directional coefficient (16) or (17), a check must be made. For the choice (16) or (17), is the slope of the augmented functional $\tilde{J}_\alpha(0)$ negative? In other words, does the descent property of the gradient phase hold?

Concerning Case (i), two subcases are possible:

$$\text{(iii)} \qquad \gamma = 0\,, \qquad \tilde{J}_\alpha(0) < 0\,, \tag{18}$$

$$\text{(iv)} \qquad \gamma = 0\,, \qquad \tilde{J}_\alpha(0) \geq 0\,. \tag{19}$$

In Case (iii), the directional coefficient (16) is accepted, and the algorithm completes the ordinary gradient phase. In Case (iv), the descent property of the ordinary gradient phase does not hold, and the value of the augmented functional cannot be reduced, owing to numerical inaccuracies; hence, this constitutes a nonconvergence condition for the algorithm as a whole.

Concerning Case (ii), two subcases are possible:

$$\text{(v)} \qquad \gamma = Q/\hat{Q}\,, \qquad \tilde{J}_\alpha(0) < 0\,, \tag{20}$$

$$\text{(vi)} \qquad \gamma = Q/\hat{Q}\,, \qquad \tilde{J}_\alpha(0) \geq 0\,. \tag{21}$$

In Case (v), the directional coefficient (17) is accepted, and the algorithm completes the conjugate gradient phase. In Case (vi), the directional coefficient (17) is rejected, and is replaced by the directional coefficient (16). This means that the algorithm is restarted with an ordinary gradient phase, characterized by $\gamma = 0$.

2.5. Conjugate Gradient Stepsize. After the directional coefficient has been selected, the functions (15) are known, and the one-parameter family of solutions (6) can be formed. Then, the conjugate gradient stepsize $\alpha$ must be determined through a one-dimensional search on the augmented functional $\tilde{J}(\alpha)$ in such a way that the following inequalities are satisfied:

$$\tilde{J}_\alpha^2(\alpha)/\tilde{J}_\alpha^2(0) \leq E\text{-}06 \tag{22}$$

and

$$\tilde{J}(\alpha) < \tilde{J}(0) \ , \tag{23}$$

$$\tilde{P}(\alpha) \leq 10 \ , \tag{24}$$

$$\tilde{\tau}(\alpha) \geq 0 \ . \tag{25}$$

Note that Ineq. (24) prevents the constraint error P from becoming too large during the conjugate gradient phase.

Scanning Process. We consider the sequence of step-sizes

$$\{\alpha\} = \{0, 1, 2, 4, 8, \ldots\} . \tag{26}$$

For every element of the sequence (26), we compute the augmented functional $\tilde{J}(\alpha)$ and its slope $\tilde{J}_\alpha(\alpha)$. Simultaneously, we monitor the constraint error $\tilde{P}(\alpha)$ and the final time $\tilde{\tau}(\alpha)$.

We denote by $\alpha_1$ and $\alpha_2$ the smallest consecutive elements in the sequence (26) such that the following inequalities are satisfied:

$$\tilde{J}_\alpha(\alpha_1) < 0 , \qquad \tilde{J}_\alpha(\alpha_2) > 0 , \tag{27}$$

simultaneously with (24)-(25). This means that a relative minimum of $\tilde{J}(\alpha)$ occurs for a value $\alpha_0$ such that

$$\alpha_1 < \alpha_0 < \alpha_2 . \tag{28}$$

Cubic Interpolation Process. Having bracketed the minimum point, we approximate the function $\tilde{J}(\alpha)$ and its derivative $\tilde{J}_\alpha(\alpha)$ as follows:

$$\tilde{J}(\alpha) = k_0 + k_1\alpha + k_2\alpha^2 + k_3\alpha^3 , \tag{29}$$

$$\tilde{J}_\alpha(\alpha) = k_1 + 2k_2\alpha + 3k_3\alpha^2 , \tag{30}$$

and determine the coefficients $k_i$ so as to satisfy the exact values of the ordinate $\tilde{J}(\alpha)$ and the slope $\tilde{J}_\alpha(\alpha)$ at $\alpha_1$ and $\alpha_2$. With the coefficients known, the optimum value of $\alpha$ is computed from one of these relations:

$$\alpha_0 = (1/3k_3)\left[-k_2 + \sqrt{(k_2^2 - 3k_1k_3)}\right], \qquad k_3^2 > E\text{-}06, \tag{31}$$

$$\alpha_0 = -k_1/2k_2, \qquad k_3^2 \leq E\text{-}06. \tag{32}$$

Here, the parameter $k_3^2$ governs the switch from cubic interpolation to quadratic interpolation.

With $\alpha_0$ known, two possibilities arise, depending on the sign of $\tilde{J}_\alpha(\alpha)$ at the point $\alpha_0$:

$$\text{(i)} \qquad \tilde{J}_\alpha(\alpha_0) > 0, \tag{33}$$

$$\text{(ii)} \qquad \tilde{J}_\alpha(\alpha_0) < 0. \tag{34}$$

In Case (i), the cubic interpolation process is repeated between $\alpha_1$ and $\alpha_0$. In Case (ii), it is repeated between $\alpha_0$ and $\alpha_2$. The process is continued iteratively until Ineq. (22) is satisfied.

Safeguards. The one-dimensional search scheme described previously requires several safeguards.

(a) After determining the optimum stepsize $\alpha_0$ in such

a way that Ineq. (22) is satisfied, Ineqs. (23)-(25) must be checked. If (23)-(25) are satisfied, the stepsize $\alpha_0$ is accepted. If (23)-(25) are violated, a bisection process is undertaken, starting from

$$\alpha = \alpha_0 , \tag{35}$$

until satisfaction of (23)-(25) occurs.

(b) Within the cubic interpolation process, let $N_{ci}$ denote the number of iterations required to satisfy Ineq. (22). The following upper limit is imposed on $N_{ci}$:

$$N_{ci} \leq 10 . \tag{36}$$

Any violation of Ineq. (36) is indicative of failure of the cubic interpolation process to produce the minimum point of the function $\tilde{J}(\alpha)$ in a reasonable number of iterations; hence, the cubic interpolation process is stopped, even though Ineq. (22) is not satisfied. However, the algorithm is not stopped.

Let $\alpha_*$ denote the suboptimal stepsize obtained when the limit (36) is reached. Once more, Ineqs. (23)-(25) must be checked. If (23)-(25) are satisfied, the stepsize $\alpha_*$ is accepted. If (23)-(25) are violated, a bisection process is undertaken, starting from

$$\alpha = \alpha_* , \tag{37}$$

until satisfaction of (23)-(25) occurs.

(c) Suppose that, in the course of the scanning process, a stepsize $\alpha_{**}$ is found such that Ineq. (24) and/or Ineq. (25) are violated prior to (or simultaneously with) attaining the bracketing conditions (27). Under these conditions, the cubic interpolation process is bypassed, and a bisection process, starting from

$$\alpha = \alpha_{**} , \tag{38}$$

is undertaken, until satisfaction of (23)-(25) occurs.

(d) Should the bisection process described in (a), (b), or (c) occur, this would have the following implication: the optimum stepsize of the present conjugate gradient phase cannot be employed, owing to violation of Ineq. (22). Because of the ensuing large violations of the orthogonality and conjugacy conditions, the directional coefficient of the next conjugate gradient phase must be reset at the level

$$\gamma = 0 , \tag{39}$$

characteristic of an ordinary gradient phase. In other words, after the restoration phase is completed, the algorithm must be restarted with an ordinary gradient phase.

(e) Regardless of whether Case (a), (b), or (c) arises, let $N_{bg}$ denote the number of bisections of the conjugate gradient stepsize required to satisfy Ineqs. (23)-(25). The following upper limit is imposed on $N_{bg}$ :

$$N_{bg} \leq 10. \tag{40}$$

Any violation of Ineq. (40) indicates extreme smallness of the conjugate gradient displacements. In this connection, two possibilities arise:

$$\text{(i)} \qquad N_{bg} > 10 \ , \qquad \gamma = 0 \ , \tag{41}$$

$$\text{(ii)} \qquad N_{bg} > 10 \ , \qquad \gamma = Q/\hat{Q} \ . \tag{42}$$

Case (i) constitutes a nonconvergence condition of the algorithm as a whole. Case (ii) constitutes a restarting condition: the directional coefficient of the present conjugate gradient phase must be reset at the level

$$\gamma = 0 \ , \tag{43}$$

characteristic of an ordinary gradient phase.

2.6. <u>Conjugate Gradient Restoration Cycle</u>. Generally speaking, the first cycle of the algorithm is a half cycle, in that it includes a restoration phase only. Every subsequent cycle is a complete cycle, in that it includes both a conjugate gradient phase and a restoration phase.

Between the endpoints of a complete conjugate gradient-restoration cycle, the following descent property must be satisfied:

$$I_3 < I_1 , \tag{44}$$

where $I_1$ denotes the value of the functional I at the beginning of the cycle and $I_3$ denotes the value of I at the end of the cycle.

If Ineq. (44) holds, the next conjugate gradient-restoration cycle can be started. If Ineq. (44) is violated, one must return to the previous conjugate gradient phase and bisect the conjugate gradient stepsize until, after restoration, Ineq. (44) is satisfied.

Safeguards. Within each conjugate gradient-restoration cycle, let $N_{bc}$ denote the number of bisections of the conjugate gradient stepsize required to satisfy Ineq. (44). If

$$N_{bc} \geq 1 , \qquad \gamma = 0 \quad \text{or} \quad \gamma = Q/\hat{Q} , \tag{45}$$

large violations of the orthogonality and conjugacy conditions ensue; hence, the directional coefficient of the next conjugate gradient phase must be reset at the level

$$\gamma = 0 , \tag{46}$$

characteristic of an ordinary gradient phase.

The following upper limit must be imposed on $N_{bc}$:

$$N_{bc} \leq 10 \ . \tag{47}$$

Any violation of Ineq. (47) indicates extreme smallness of the conjugate gradient displacements. In this connection, two possibilities arise:

$$\text{(i)} \qquad N_{bc} > 10 \ , \qquad \gamma = 0 \ , \tag{48}$$

$$\text{(ii)} \qquad N_{bc} > 10 \ , \qquad \gamma = Q/\hat{Q} \ . \tag{49}$$

Case (i) constitutes a nonconvergence condition of the algorithm as a whole. Case (ii) constitutes a restarting condition: the directional coefficient of the present conjugate gradient phase must be reset at the level

$$\gamma = 0 \ , \tag{50}$$

characteristic of an ordinary gradient phase.

2.7. Summary of Conditions. From the previous discussion, the convergence conditions can be summarized as follows:

$$\text{(i)} \qquad P \leq E\text{-}08 \ , \tag{51-1}$$

$$\text{(ii)} \qquad P \leq E\text{-}08 \ , \qquad Q \leq E\text{-}04 \ . \tag{51-2}$$

Here, Ineq. (51-1) applies to the restoration phase; and Ineqs. (51-2) apply to the algorithm as a whole.

The nonconvergence conditions can be summarized as

follows:

$$\text{(i)} \quad N > 100\,, \tag{52}$$

$$\text{(ii)} \quad N_r > 10\,, \tag{53}$$

$$\text{(iii)} \quad N_{br} > 10\,, \tag{54}$$

$$\text{(iv)} \quad \tilde{J}_\alpha(0) \geq 0\,, \qquad \gamma = 0\,, \tag{55}$$

$$\text{(v)} \quad N_{bg} > 10\,, \qquad \gamma = 0\,, \tag{56}$$

$$\text{(vi)} \quad N_{bc} > 10\,, \qquad \gamma = 0\,, \tag{57}$$

$$\text{(vii)} \quad M > 0.83E+75\,. \tag{58}$$

Here, (52) applies to the algorithm as a whole; (53)-(54) apply to the restoration phase; (55)-(57) apply to the conjugate gradient phase; and (58) indicates overflow: the modulus M of some of the quantities used in the algorithm has reached the upper limit allowed by the IBM 370/155 computer.

Finally, the restarting conditions can be summarized as follows:

$$\text{(i)} \quad \tilde{J}_\alpha(0) \geq 0\,, \qquad \gamma = Q/\hat{Q}\,, \tag{59}$$

$$\text{(ii)} \quad N_{bg} \geq 1\,, \qquad \gamma = 0 \quad \text{or} \quad \gamma = Q/\hat{Q}\,, \tag{60}$$

$$\text{(iii)} \quad N_{bc} \geq 1\,, \qquad \gamma = 0 \quad \text{or} \quad \gamma = Q/\hat{Q}\,, \tag{61}$$

$$\text{(iv)} \quad N_{bg} > 10\,, \qquad \gamma = Q/\hat{Q}\,, \tag{62}$$

$$\text{(v)} \quad N_{bc} > 10\,, \qquad \gamma = Q/\hat{Q}\,. \tag{63}$$

All of these relations apply to the conjugate gradient phase. Specifically, (59),(62),(63) require restarting with $\gamma = 0$ in the present conjugate gradient phase; and (60)-(61) require starting with $\gamma = 0$ in the next conjugate gradient phase.

## 3. Numerical Examples: Linear-Quadratic Case

In this section, four numerical examples involving quadratic functionals and linear constraints are described. For simplicity, the symbols employed here denote scalar quantities.[6]

Example 3.1. Consider the following linear-quadratic problem (LQ-problem) with free end coordinates (Ref. 8 ):

$$I = \int_0^1 (x^2 + y^2 + u^2/400 + v^2/400)\,dt, \tag{64}$$

$$\dot{x} = y, \qquad \dot{y} = (u + v)/2 - y, \tag{65}$$

$$u - v = 0, \tag{66}$$

$$x(0) = 0, \qquad y(0) = -1. \tag{67}$$

Assume the nominal functions

$$x(t) = 0, \qquad y(t) = -1, \tag{68}$$

$$u(t) = 0, \qquad v(t) = 0. \tag{69}$$

The results for the sequential conjugate gradient-restoration

[6] The symbols $x(t)$, $y(t)$ denote the components of the state. The symbols $u(t)$, $v(t)$ denote the components of the control.

algorithm (SCGRA) of Ref. 1 are given in Tables 1-3. The results for the sequential ordinary gradient-restoration algorithm (SOGRA) of Ref. 2 are given in Tables 4-6. Note that SCGRA leads to the solution in $N = 4$ iterations, while SOGRA leads to the solution in $N = 8$ iterations.

Table 1. Convergence history, SCGRA, Example 3.1.

| N | Phase | $\gamma$ | P | Q | I |
|---|---|---|---|---|---|
| 0 | | | 0.20 E+01 | | |
| 1 | REST | | 0.00 E-30 | 0.20 E+00 | 0.60042 |
| 2 | GRAD | $\gamma = 0$ | 0.72 E-31 | 0.66 E-02 | 0.16135 |
| 3 | GRAD | $\gamma \neq 0$ | 0.18 E-30 | 0.58 E-03 | 0.08794 |
| 4 | GRAD | $\gamma \neq 0$ | 0.19 E-30 | 0.60 E-04 | 0.07216 |

Table 2. Converged state variables, SCGRA, Example 3.1.

| t | x | y |
|---|---|---|
| 0.0 | 0.0000 | -1.0000 |
| 0.1 | -0.0568 | -0.2461 |
| 0.2 | -0.0651 | 0.0239 |
| 0.3 | -0.0595 | 0.0662 |
| 0.4 | -0.0544 | 0.0308 |
| 0.5 | -0.0533 | -0.0063 |
| 0.6 | -0.0548 | -0.0180 |
| 0.7 | -0.0561 | -0.0053 |
| 0.8 | -0.0555 | 0.0177 |
| 0.9 | -0.0527 | 0.0358 |
| 1.0 | -0.0488 | 0.0395 |

Table 3. Converged control variables, SCGRA, Example 3.1.

| t | u | v |
|---|---|---|
| 0.0 | 10.1518 | 10.1518 |
| 0.1 | 4.3400 | 4.3400 |
| 0.2 | 1.2590 | 1.2590 |
| 0.3 | -0.0776 | -0.0776 |
| 0.4 | -0.4104 | -0.4104 |
| 0.5 | -0.2670 | -0.2670 |
| 0.6 | 0.0036 | 0.0036 |
| 0.7 | 0.2017 | 0.2017 |
| 0.8 | 0.2466 | 0.2466 |
| 0.9 | 0.1517 | 0.1517 |
| 1.0 | 0.0000 | 0.0000 |
| | $\tau = 1.0000$ | |

Table 4. Convergence history, SOGRA, Example 3.1.

| N | Phase | P | Q | I |
|---|---|---|---|---|
| 0 | | 0.20 E+01 | | |
| 1 | REST | 0.00 E-30 | 0.20 E+00 | 0.60042 |
| 2 | GRAD | 0.72 E-31 | 0.66 E-02 | 0.16135 |
| 3 | GRAD | 0.12 E-30 | 0.46 E-02 | 0.09845 |
| 4 | GRAD | 0.15 E-30 | 0.57 E-03 | 0.08763 |
| 5 | GRAD | 0.22 E-30 | 0.15 E-02 | 0.08129 |
| 6 | GRAD | 0.23 E-30 | 0.21 E-03 | 0.07752 |
| 7 | GRAD | 0.27 E-30 | 0.67 E-03 | 0.07509 |
| 8 | GRAD | 0.27 E-30 | 0.95 E-04 | 0.07350 |

Table 5. Converged state variables, SOGRA, Example 3.1.

| t | x | y |
|---|---|---|
| 0.0 | 0.0000 | -1.0000 |
| 0.1 | -0.0607 | -0.3093 |
| 0.2 | -0.0753 | -0.0294 |
| 0.3 | -0.0729 | 0.0570 |
| 0.4 | -0.0665 | 0.0645 |
| 0.5 | -0.0608 | 0.0482 |
| 0.6 | -0.0569 | 0.0307 |
| 0.7 | -0.0545 | 0.0184 |
| 0.8 | -0.0530 | 0.0107 |
| 0.9 | -0.0522 | 0.0059 |
| 1.0 | -0.0518 | 0.0036 |

Table 6. Converged control variables, SOGRA, Example 3.1.

| t | u | v |
|---|---|---|
| 0.0 | 9.0622 | 9.0622 |
| 0.1 | 4.0677 | 4.0677 |
| 0.2 | 1.5440 | 1.5440 |
| 0.3 | 0.3966 | 0.3966 |
| 0.4 | -0.0372 | -0.0372 |
| 0.5 | -0.1409 | -0.1409 |
| 0.6 | -0.1211 | -0.1211 |
| 0.7 | -0.0781 | -0.0781 |
| 0.8 | -0.0487 | -0.0487 |
| 0.9 | -0.0318 | -0.0318 |
| 1.0 | 0.0000 | 0.0000 |

$\tau = 1.0000$

Example 3.2. Consider the following LQ-problem with free end coordinates:

$$I = \int_0^1 (x^2 + y^2 + u^2/200 + v^2)\,dt, \tag{70}$$

$$\dot{x} = y, \qquad \dot{y} = u + v - y, \tag{71}$$

$$u + 2v - 10/(1 + 10t)^2 = 0, \tag{72}$$

$$x(0) = 0, \qquad y(0) = -1. \tag{73}$$

Assume the nominal functions

$$x(t) = 0, \qquad y(t) = -1, \tag{74}$$

$$u(t) = 0, \qquad v(t) = 0. \tag{75}$$

The results for the sequential conjugate-gradient restoration algorithm (SCGRA) of Ref. 1 are given in Tables 7-9. The results for the sequential ordinary gradient-restoration algorithm of Ref. 2 are given in Tables 10-12. Note that both SCGRA and SOGRA lead to the solution in $N = 3$ iterations.

Table 7. Convergence history, SCGRA, Example 3.2.

| N | Phase | $\gamma$ | P | Q | I |
|---|---|---|---|---|---|
| 0 | | | 0.53 E+01 | | |
| 1 | REST | | 0.19 E-31 | 0.58 E+00 | 0.73349 |
| 2 | GRAD | $\gamma = 0$ | 0.15 E-30 | 0.52 E-02 | 0.09642 |
| 3 | GRAD | $\gamma \neq 0$ | 0.27 E-30 | 0.24 E-05 | 0.09066 |

Table 8. Converged state variables,SCGRA, Example 3.2.

| t | x | y |
|---|---|---|
| 0.0 | 0.0000 | -1.0000 |
| 0.1 | -0.0652 | -0.4282 |
| 0.2 | -0.0966 | -0.2249 |
| 0.3 | -0.1134 | -0.1214 |
| 0.4 | -0.1223 | -0.0608 |
| 0.5 | -0.1263 | -0.0228 |
| 0.6 | -0.1273 | 0.0018 |
| 0.7 | -0.1262 | 0.0180 |
| 0.8 | -0.1238 | 0.0288 |
| 0.9 | -0.1206 | 0.0361 |
| 1.0 | -0.1167 | 0.0410 |

Table 9. Converged control variables, SCGRA, Example 3.2.

| t | u | v |
|---|---|---|
| 0.0 | 10.0894 | -0.0447 |
| 0.1 | 2.6274 | -0.0637 |
| 0.2 | 1.1967 | -0.0428 |
| 0.3 | 0.6741 | -0.0245 |
| 0.4 | 0.4222 | -0.0111 |
| 0.5 | 0.2814 | -0.0018 |
| 0.6 | 0.1963 | 0.0038 |
| 0.7 | 0.1429 | 0.0066 |
| 0.8 | 0.1096 | 0.0069 |
| 0.9 | 0.0901 | 0.0049 |
| 1.0 | 0.0808 | 0.0009 |

$\tau = 1.0000$

Table 10. Convergence history, SOGRA, Example 3.2.

| N | Phase | P | Q | I |
|---|---|---|---|---|
| 0 | | 0.53 E+01 | | |
| 1 | REST | 0.19 E-31 | 0.58 E+00 | 0.73349 |
| 2 | GRAD | 0.15 E-30 | 0.52 E-02 | 0.09642 |
| 3 | GRAD | 0.24 E-30 | 0.49 E-04 | 0.09072 |

Table 11. Converged state variables, SOGRA, Example 3.2.

| t | x | y |
|---|---|---|
| 0.0 | 0.0000 | -1.0000 |
| 0.1 | -0.0653 | -0.4300 |
| 0.2 | -0.0969 | -0.2270 |
| 0.3 | -0.1139 | -0.1237 |
| 0.4 | -0.1230 | -0.0630 |
| 0.5 | -0.1273 | -0.0249 |
| 0.6 | -0.1285 | -0.0002 |
| 0.7 | -0.1276 | 0.0161 |
| 0.8 | -0.1254 | 0.0271 |
| 0.9 | -0.1223 | 0.0344 |
| 1.0 | -0.1186 | 0.0395 |

Table 12. Converged control variables, SOGRA, Example 3.2.

| t | u | v |
|---|---|---|
| 0.0 | 10.0170 | -0.0085 |
| 0.1 | 2.6084 | -0.0542 |
| 0.2 | 1.1880 | -0.0384 |
| 0.3 | 0.6692 | -0.0221 |
| 0.4 | 0.4191 | -0.0095 |
| 0.5 | 0.2794 | -0.0008 |
| 0.6 | 0.1949 | 0.0045 |
| 0.7 | 0.1419 | 0.0071 |
| 0.8 | 0.1088 | 0.0072 |
| 0.9 | 0.0895 | 0.0052 |
| 1.0 | 0.0802 | 0.0012 |

$\tau = 1.0000$

Example 3.3. Consider the following LQ-problem with final coordinates partly given and partly free:

$$I = \int_0^1 (x^2 + y^2 + u^2/400 + v^2/400)dt, \tag{76}$$

$$\dot{x} = y, \qquad \dot{y} = (u+v)/2 - y, \tag{77}$$

$$u - v = 0, \tag{78}$$

$$x(0) = 0, \qquad y(0) = -1, \tag{79}$$

$$x(1) = 0. \tag{80}$$

Assume the nominal functions

$$x(t) = 0, \qquad y(t) = -1 + t, \tag{81}$$

$$u(t) = 0, \qquad v(t) = 0. \tag{82}$$

The results for the sequential conjugate gradient-restoration algorithm (SCGRA) of Ref. 1 are given in Tables 13-15. The results for the sequential ordinary gradient-restoration algorithm (SOGRA) of Ref. 2 are given in Tables 16-18. Note that SCGRA leads to the solution in $N = 3$ iterations, while SOGRA leads to the solution in $N = 4$ iterations.

Table 13. Convergence history, SCGRA, Example 3.3.

| N | Phase | $\gamma$ | P | Q | I |
|---|---|---|---|---|---|
| 0 | | | 0.66 E+00 | | |
| 1 | REST | | 0.18 E-30 | 0.96 E-02 | 0.20082 |
| 2 | GRAD | $\gamma = 0$ | 0.60 E-29 | 0.67 E-03 | 0.09395 |
| 3 | GRAD | $\gamma \neq 0$ | 0.13 E-27 | 0.70 E-04 | 0.07580 |

Table 14. Converged state variables, SCGRA, Example 3.3.

| t | x | y |
|---|---|---|
| 0.0 | 0.0000 | -1.0000 |
| 0.1 | -0.0551 | -0.2146 |
| 0.2 | -0.0594 | 0.0702 |
| 0.3 | -0.0489 | 0.1167 |
| 0.4 | -0.0387 | 0.0809 |
| 0.5 | -0.0327 | 0.0426 |
| 0.6 | -0.0292 | 0.0316 |
| 0.7 | -0.0254 | 0.0478 |
| 0.8 | -0.0193 | 0.0765 |
| 0.9 | -0.0103 | 0.0998 |
| 1.0 | 0.0000 | 0.1031 |

Table 15. Converged control variables, SCGRA, Example 3.3.

| t | u | v |
|---|---|---|
| 0.0 | 10.5662 | 10.5662 |
| 0.1 | 4.5935 | 4.5935 |
| 0.2 | 1.3902 | 1.3902 |
| 0.3 | -0.0168 | -0.0168 |
| 0.4 | -0.3736 | -0.3736 |
| 0.5 | -0.2220 | -0.2220 |
| 0.6 | 0.0743 | 0.0743 |
| 0.7 | 0.3019 | 0.3019 |
| 0.8 | 0.3654 | 0.3654 |
| 0.9 | 0.2534 | 0.2534 |
| 1.0 | 0.0000 | 0.0000 |

$\tau = 1.0000$

Table 16. Convergence history, SOGRA, Example 3.3.

| N | Phase | P | Q | I |
|---|---|---|---|---|
| 0 | | 0.66 E+00 | | |
| 1 | REST | 0.18 E-30 | 0.96 E-02 | 0.20082 |
| 2 | GRAD | 0.60 E-29 | 0.67 E-03 | 0.09395 |
| 3 | GRAD | 0.10 E-27 | 0.25 E-03 | 0.07843 |
| 4 | GRAD | 0.19 E-27 | 0.56 E-04 | 0.07515 |

Table 17. Converged state variables, SOGRA, Example 3.3.

| t | x | y |
|---|---|---|
| 0.0 | 0.0000 | -1.0000 |
| 0.1 | -0.0556 | -0.2280 |
| 0.2 | -0.0617 | 0.0528 |
| 0.3 | -0.0521 | 0.1180 |
| 0.4 | -0.0405 | 0.1072 |
| 0.5 | -0.0311 | 0.0826 |
| 0.6 | -0.0237 | 0.0654 |
| 0.7 | -0.0176 | 0.0585 |
| 0.8 | -0.0118 | 0.0583 |
| 0.9 | -0.0059 | 0.0598 |
| 1.0 | 0.0000 | 0.0579 |

Table 18. Converged control variables, SOGRA, Example 3.3.

| t | u | v |
|---|---|---|
| 0.0 | 10.5915 | 10.5915 |
| 0.1 | 4.4265 | 4.4265 |
| 0.2 | 1.4605 | 1.4605 |
| 0.3 | 0.2376 | 0.2376 |
| 0.4 | -0.1294 | -0.1294 |
| 0.5 | -0.1414 | -0.1414 |
| 0.6 | -0.0518 | -0.0518 |
| 0.7 | 0.0311 | 0.0311 |
| 0.8 | 0.0734 | 0.0734 |
| 0.9 | 0.0673 | 0.0673 |
| 1.0 | 0.0000 | 0.0000 |

$\tau = 1.0000$

Example 3.4. Consider the following LQ-problem with final coordinates partly given and partly free:

$$I = \int_0^1 (x^2 + y^2 + u^2 + v^2)dt, \tag{33}$$

$$\dot{x} = y, \qquad \dot{y} = u - y, \tag{84}$$

$$u - v + 5t - 3 = 0, \tag{85}$$

$$x(0) = 0, \qquad y(0) = -1\ , \tag{86}$$

$$x(1) = 0. \tag{87}$$

Assume the nominal functions

$$x(t) = 0, \qquad y(t) = -1+t, \tag{88}$$

$$u(t) = 0, \qquad v(t) = 0. \tag{89}$$

The results for the sequential conjugate gradient-restoration algorithm (SCGRA) of Ref. 1 are given in Tables 19-21. The results for the sequential ordinary gradient-restoration algorithm (SOGRA) of Ref. 2 are given in Tables 22-24. Note that both SCGRA and SOGRA lead to the solution in $N = 2$ iterations.

Table 19. Convergence history, SCGRA, Example 3.4.

| N | Phase | $\gamma$ | P | Q | I |
|---|---|---|---|---|---|
| 0 | | | 0.30 E+01 | | |
| 1 | REST | | 0.29 E-29 | 0.54 E-02 | 3.31804 |
| 2 | GRAD | $\gamma = 0$ | 0.13 E-30 | 0.29 E-06 | 3.31669 |

Table 20. Converged state variables, SCGRA, Example 3.4.

| t | x | y |
|---|---|---|
| 0.0 | 0.0000 | -1.0000 |
| 0.1 | -0.0806 | -0.6263 |
| 0.2 | -0.1276 | -0.3251 |
| 0.3 | -0.1478 | -0.0892 |
| 0.4 | -0.1475 | 0.0874 |
| 0.5 | -0.1321 | 0.2102 |
| 0.6 | -0.1071 | 0.2833 |
| 0.7 | -0.0770 | 0.3106 |
| 0.8 | -0.0464 | 0.2949 |
| 0.9 | -0.0194 | 0.2386 |
| 1.0 | 0.0000 | 0.1433 |

Table 21. Converged control variables, SCGRA, Example 3.4.

| t | u | v |
|---|---|---|
| 0.0 | 3.1238 | 0.1238 |
| 0.1 | 2.7352 | 0.2352 |
| 0.2 | 2.3497 | 0.3497 |
| 0.3 | 1.9645 | 0.4645 |
| 0.4 | 1.5766 | 0.5766 |
| 0.5 | 1.1828 | 0.6828 |
| 0.6 | 0.7798 | 0.7798 |
| 0.7 | 0.3640 | 0.8640 |
| 0.8 | -0.0683 | 0.9316 |
| 0.9 | -0.5216 | 0.9783 |
| 1.0 | -1.0000 | 1.0000 |

$\tau = 1.0000$

Table 22. Convergence history, SOGRA, Example 3.4.

| N | Phase | P | Q | I |
|---|---|---|---|---|
| 0 | | 0.30 E+01 | | |
| 1 | REST | 0.29 E-29 | 0.54 E-02 | 3.31804 |
| 2 | GRAD | 0.13 E-30 | 0.29 E-06 | 3.31669 |

Table 23. Converged state variables, SOGRA, Example 3.4.

| t | x | y |
|---|---|---|
| 0.0 | 0.0000 | -1.0000 |
| 0.1 | -0.0806 | -0.6263 |
| 0.2 | -0.1276 | -0.3251 |
| 0.3 | -0.1478 | -0.0892 |
| 0.4 | -0.1475 | 0.0874 |
| 0.5 | -0.1321 | 0.2102 |
| 0.6 | -0.1071 | 0.2833 |
| 0.7 | -0.0770 | 0.3106 |
| 0.8 | -0.0464 | 0.2949 |
| 0.9 | -0.0194 | 0.2386 |
| 1.0 | 0.0000 | 0.1433 |

Table 24. Converged control variables, SOGRA, Example 3.4.

| t | u | v |
|---|---|---|
| 0.0 | 3.1238 | 0.1238 |
| 0.1 | 2.7352 | 0.2352 |
| 0.2 | 2.3497 | 0.3497 |
| 0.3 | 1.9645 | 0.4645 |
| 0.4 | 1.5766 | 0.5766 |
| 0.5 | 1.1828 | 0.6828 |
| 0.6 | 0.7798 | 0.7798 |
| 0.7 | 0.3640 | 0.8640 |
| 0.8 | -0.0683 | 0.9316 |
| 0.9 | -0.5216 | 0.9783 |
| 1.0 | -1.0000 | 1.0000 |

$\tau = 1.0000$

## 4. Numerical Examples: Nonlinear Nonquadratic Case

In this section, twelve numerical examples involving nonquadratic functionals and nonlinear constraints are described. These examples include bounds on the state or the time rate of change of the state. They can be converted into the model of Ref. 1 by means of the transformation techniques outlined in Refs. 9-10. For simplicity, the symbols employed here denote scalar quantities.[7]

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[7] The symbols x(t), y(t), z(t), h(t) denote the components of the state. The symbols u(t), v(t), w(t) denote the components of the control. The symbol $\tau$ denotes the parameter (the unknown final time, for those problems where the final time is free).

Example 4.1. Consider the following problem with bounded state and fixed endpoints:

$$I = \int_0^1 (x^2 + u^2)dt, \tag{90}$$

$$\dot{x} = x^2 - u, \tag{91}$$

$$x - 0.9 \geq 0, \tag{92}$$

$$x(0) = 1, \tag{93}$$

$$x(1) = 1. \tag{94}$$

By means of the transformation techniques of Ref. 9 , the previous problem can be converted into the following equality constrained problem:

$$I = \int_0^1 (x^2 + u^2)dt, \tag{95}$$

$$\dot{x} = x^2 - u, \qquad \dot{y} = v, \tag{96}$$

$$x^2 - u - 2yv = 0, \tag{97}$$

$$x(0) = 1, \qquad y(0) = \sqrt{(0.1)}, \tag{98}$$

$$x(1) = 1. \tag{99}$$

Assume the nominal functions

$$x(t) = 1, \qquad y(t) = \sqrt{(0.1)}, \tag{100}$$

$$u(t) = 1, \qquad v(t) = 1. \tag{101}$$

The results for the sequential conjugate gradient-restoration algorithm (SCGRA) of Ref. 1 are given in Tables 25-27. The results for the sequential ordinary gradient-restoration algorithm (SOGRA) of Ref. 2 are given in Tables 28-30. Note that SCGRA leads to the solution in $N = 14$ iterations, while SOGRA leads to the solution in $N = 12$ iterations.

Table 25. Convergence history, SCGRA, Example 4.1.

| N | Phase | $\gamma$ | P | Q | I |
|---|---|---|---|---|---|
| 0 | | | 0.14 E+01 | | |
| 1 | REST | | 0.33 E-02 | | |
| 2 | REST | | 0.12 E-04 | | |
| 3 | REST | | 0.52 E-09 | 0.35 E+00 | 1.83569 |
| 4 | GRAD | $\gamma = 0$ | 0.68 E-02 | | |
| 5 | REST | | 0.11 E-05 | | |
| 6 | REST | | 0.15 E-16 | 0.14 E-01 | 1.66599 |
| 7 | GRAD | $\gamma \neq 0$ | 0.22 E-03 | | |
| 8 | REST | | 0.12 E-08 | 0.27 E-03 | 1 65745 |
| 9 | GRAD | $\gamma \neq 0$ | 0.26 E-05 | | |
| 10 | REST | | 0.12 E-12 | 0.18 E-03 | 1.65704 |
| 11 | GRAD | $\gamma \neq 0$ | 0.17 E-05 | | |
| 12 | REST | | 0.51 E-13 | 0.16 E-03 | 1.65677 |
| 13 | GRAD | $\gamma \neq 0$ | 0.66 E-05 | | |
| 14 | REST | | 0.34 E-12 | 0.66 E-04 | 1.65641 |

Table 26. Converged state variables, SCGRA, Example 4.1.

| t | x | y |
|---|---|---|
| 0.0 | 1.0000 | 0.3162 |
| 0.1 | 0.9417 | 0.2043 |
| 0.2 | 0.9082 | 0.0907 |
| 0.3 | 0.9002 | 0.0151 |
| 0.4 | 0.9000 | -0.0027 |
| 0.5 | 0.9000 | -0.0013 |
| 0.6 | 0.9000 | -0.0010 |
| 0.7 | 0.9003 | 0.0175 |
| 0.8 | 0.9087 | 0.0933 |
| 0.9 | 0.9420 | 0.2051 |
| 1.0 | 1.0000 | 0.3162 |

Table 27. Converged control variables, SCGRA, Example 4.1.

| t | u | v |
|---|---|---|
| 0.0 | 1.7514 | -1.1881 |
| 0.1 | 1.3437 | -1.1180 |
| 0.2 | 1.0166 | -1.0558 |
| 0.3 | 0.8232 | -0.4215 |
| 0.4 | 0.8099 | -0.0180 |
| 0.5 | 0.8100 | 0.0096 |
| 0.6 | 0.8100 | 0.0303 |
| 0.7 | 0.7956 | 0.4250 |
| 0.8 | 0.6292 | 1.0524 |
| 0.9 | 0.4398 | 1.0912 |
| 1.0 | 0.2376 | 1.2053 |

$\tau = 1.0000$

Table 28. Convergence history, SOGRA, Example 4.1.

| N | Phase | P | Q | I |
|---|---|---|---|---|
| 0 | | 0.14 E+01 | | |
| 1 | REST | 0.33 E-02 | | |
| 2 | REST | 0.12 E-04 | | |
| 3 | REST | 0.52 E-09 | 0.35 E+00 | 1.83569 |
| 4 | GRAD | 0.68 E-02 | | |
| 5 | REST | 0.11 E-05 | | |
| 6 | REST | 0.15 E-16 | 0.14 E-01 | 1.66599 |
| 7 | GRAD | 0.67 E-04 | | |
| 8 | REST | 0.10 E-09 | 0.24 E-03 | 1.65742 |
| 9 | GRAD | 0.10 E-06 | | |
| 10 | REST | 0.60 E-17 | 0.15 E-03 | 1.65697 |
| 11 | GRAD | 0.34 E-07 | | |
| 12 | REST | 0.96 E-18 | 0.93 E-04 | 1.65678 |

Table 29. Converged state variables, SOGRA, Example 4.1.

| t | x | y |
|---|---|---|
| 0.0 | 1.0000 | 0.3162 |
| 0.1 | 0.9410 | 0.2025 |
| 0.2 | 0.9095 | 0.0978 |
| 0.3 | 0.9006 | 0.0246 |
| 0.4 | 0.9000 | -0.0090 |
| 0.5 | 0.9003 | -0.0177 |
| 0.6 | 0.9000 | -0.0094 |
| 0.7 | 0.9005 | 0.0238 |
| 0.8 | 0.9094 | 0.0972 |
| 0.9 | 0.9409 | 0.2024 |
| 1.0 | 1.0000 | 0.3162 |

Table 30. Converged control variables, SOGRA, Example 4.1.

| t | u | v |
|---|---|---|
| 0.0 | 1.7482 | -1.1831 |
| 0.1 | 1.3353 | -1.1104 |
| 0.2 | 1.0097 | -0.9324 |
| 0.3 | 0.8366 | -0.5177 |
| 0.4 | 0.8067 | -0.1865 |
| 0.5 | 0.8104 | -0.0018 |
| 0.6 | 0.8135 | 0.1816 |
| 0.7 | 0.7864 | 0.5158 |
| 0.8 | 0.6442 | 0.9398 |
| 0.9 | 0.4360 | 1.1097 |
| 1.0 | 0.2470 | 1.1904 |

$\tau = 1.0000$

Example 4.2. Consider the following problem with bounded state and free final time:

$$I = \tau, \tag{102}$$

$$\dot{x} = \tau u, \qquad \dot{y} = \tau(u^2 - x^2), \tag{103}$$

$$0.4 - y \geq 0, \tag{104}$$

$$x(0) = 0, \qquad y(0) = 0, \tag{105}$$

$$x(1) = 1, \qquad y(1) = 0. \tag{106}$$

By means of the transformation techniques of Ref. 9 , the previous problem can be converted into the following equality constrained problem:

$$I = \tau, \tag{107}$$

$$\dot{x} = \tau u, \qquad \dot{y} = \tau(u^2 - x^2), \qquad \dot{z} = \tau v, \tag{108}$$

$$u^2 - x^2 + 2zv = 0, \tag{109}$$

$$x(0) = 0, \qquad y(0) = 0, \qquad z(0) = \sqrt{(0.4)}, \tag{110}$$

$$x(1) = 1, \qquad y(1) = 0. \tag{111}$$

Assume the nominal functions

$$x(t) = t, \qquad y(t) = 0, \qquad z(t) = \sqrt{(0.4)}, \tag{112}$$

$$u(t) = 1, \qquad v(t) = 1, \tag{113}$$

$$\tau = 1. \tag{114}$$

The results for the sequential conjugate gradient-restoration algorithm (SCGRA) of Ref. 1 are given in Tables 31-33. The results for the sequential ordinary gradient-restoration algorithm (SOGRA) of Ref. 2 are given in Tables 34-36. Note that SCGRA leads to the solution in $N = 28$ iterations, while SOGRA leads to the solution in $N = 34$ iterations.

Table 31. Convergence history, SCGRA, Example 4.2.

| N | Phase | $\gamma$ | P | Q | I |
|---|---|---|---|---|---|
| 0 | | | 0.53 E+01 | | |
| 1 | REST | | 0.30 E+00 | | |
| 2 | REST | | 0.58 E-02 | | |
| 3 | REST | | 0.89 E-05 | | |
| 4 | REST | | 0.69 E-10 | 0.10 E+00 | 1.63945 |
| 5 | GRAD | $\gamma = 0$ | 0.49 E-02 | | |
| 6 | REST | | 0.35 E-05 | | |
| 7 | REST | | 0.70 E-11 | 0.15 E-01 | 1.59570 |
| 8 | GRAD | $\gamma \neq 0$ | 0.88 E-03 | | |
| 9 | REST | | 0.13 E-06 | | |
| 10 | REST | | 0.38 E-14 | 0.52 E-02 | 1.58869 |
| 11 | GRAD | $\gamma \neq 0$ | 0.55 E-03 | | |
| 12 | REST | | 0.76 E-07 | | |
| 13 | REST | | 0.18 E-14 | 0.20 E-02 | 1.58534 |
| 14 | GRAD | $\gamma \neq 0$ | 0.16 E-03 | | |
| 15 | REST | | 0.11 E-07 | | |
| 16 | REST | | 0.39 E-16 | 0.83 E-03 | 1.58411 |
| 17 | GRAD | $\gamma = 0$ | 0.92 E-06 | | |
| 18 | REST | | 0.16 E-12 | 0.57 E-03 | 1.58381 |
| 19 | GRAD | $\gamma \neq 0$ | 0.13 E-04 | | |
| 20 | REST | | 0.27 E-10 | 0.51 E-03 | 1.58337 |
| 21 | GRAD | $\gamma = 0$ | 0.45 E-06 | | |
| 22 | REST | | 0.14 E-13 | 0.20 E-03 | 1.58322 |
| 23 | GRAD | $\gamma \neq 0$ | 0.18 E-05 | | |
| 24 | REST | | 0.25 E-12 | 0.16 E-03 | 1.58310 |
| 25 | GRAD | $\gamma \neq 0$ | 0.33 E-05 | | |
| 26 | REST | | 0.14 E-11 | 0.19 E-03 | 1.58299 |
| 27 | GRAD | $\gamma \neq 0$ | 0.75 E-05 | | |
| 28 | REST | | 0.22 E-10 | 0.86 E-04 | 1.58285 |

Table 32. Converged state variables, SCGRA, Example 4.2.

| t | x | y | z |
|---|---|---|---|
| 0.0 | 0.0000 | 0.0000 | 0.6324 |
| 0.1 | 0.1410 | 0.1246 | 0.5247 |
| 0.2 | 0.2791 | 0.2378 | 0.4026 |
| 0.3 | 0.4102 | 0.3273 | 0.2694 |
| 0.4 | 0.5308 | 0.3840 | 0.1264 |
| 0.5 | 0.6361 | 0.3998 | 0.0106 |
| 0.6 | 0.7439 | 0.3980 | 0.0438 |
| 0.7 | 0.8425 | 0.3595 | 0.2011 |
| 0.8 | 0.9190 | 0.2733 | 0.3558 |
| 0.9 | 0.9717 | 0.1491 | 0.5008 |
| 1.0 | 1.0000 | 0.0000 | 0.6324 |

Table 33. Converged control variables, SCGRA, Example 4.2.

| t | u | v |
|---|---|---|
| 0.0 | 0.8935 | -0.6311 |
| 0.1 | 0.8847 | -0.7269 |
| 0.2 | 0.8564 | -0.8139 |
| 0.3 | 0.7950 | -0.8604 |
| 0.4 | 0.7227 | -0.9510 |
| 0.5 | 0.6414 | -0.3163 |
| 0.6 | 0.6994 | 0.7327 |
| 0.7 | 0.5574 | 0.9920 |
| 0.8 | 0.4135 | 0.9464 |
| 0.9 | 0.2560 | 0.8770 |
| 1.0 | 0.1111 | 0.7808 |

$\tau = 1.5828$

Table 34. Convergence history, SOGRA, Example 4.2.

| N | Phase | P | Q | I |
|---|---|---|---|---|
| 0 | | 0.53 E+01 | | |
| 1 | REST | 0.30 E+00 | | |
| 2 | REST | 0.58 E-02 | | |
| 3 | REST | 0.89 E-05 | | |
| 4 | REST | 0.69 E-10 | 0.10 E+00 | 1.63945 |
| 5 | GRAD | 0.49 E-02 | | |
| 6 | REST | 0.35 E-05 | | |
| 7 | REST | 0.70 E-11 | 0.15 E-01 | 1.59570 |
| 8 | GRAD | 0.28 E-03 | | |
| 9 | REST | 0.17 E-07 | | |
| 10 | REST | 0.26 E-16 | 0.74 E-02 | 1.59032 |
| 11 | GRAD | 0.10 E-03 | | |
| 12 | REST | 0.30 E-09 | 0.51 E-02 | 1.58767 |
| 13 | GRAD | 0.35 E-04 | | |
| 14 | REST | 0.15 E-09 | 0.22 E-02 | 1.58625 |
| 15 | GRAD | 0.18 E-04 | | |
| 16 | REST | 0.70 E-11 | 0.25 E-02 | 1.58530 |
| 17 | GRAD | 0.98 E-05 | | |
| 18 | REST | 0.71 E-11 | 0.89 E-03 | 1.58469 |
| 19 | GRAD | 0.54 E-05 | | |
| 20 | REST | 0.53 E-12 | 0.14 E-02 | 1.58423 |
| 21 | GRAD | 0.35 E-05 | | |
| 22 | REST | 0.67 E-12 | 0.41 E-03 | 1.58391 |
| 23 | GRAD | 0.19 E-05 | | |
| 24 | REST | 0.67 E-13 | 0.86 E-03 | 1.58365 |
| 25 | GRAD | 0.14 E-05 | | |
| 26 | REST | 0.90 E-13 | 0.20 E-03 | 1.58347 |
| 27 | GRAD | 0.82 E-06 | | |
| 28 | REST | 0.11 E-13 | 0.54 E-03 | 1.58332 |
| 29 | GRAD | 0.64 E-06 | | |
| 30 | REST | 0.15 E-13 | 0.10 E-03 | 1.58321 |
| 31 | GRAD | 0.38 E-06 | | |
| 32 | REST | 0.25 E-14 | 0.36 E-03 | 1.58312 |
| 33 | GRAD | 0.31 E-06 | | |
| 34 | REST | 0.34 E-14 | 0.58 E-04 | 1.58305 |

Table 35. Converged state variables, SOGRA, Example 4.2.

| t | x | y | z |
|---|---|---|---|
| 0.0 | 0.0000 | 0.0000 | 0.6324 |
| 0.1 | 0.1414 | 0.1253 | 0.5240 |
| 0.2 | 0.2793 | 0.2382 | 0.4022 |
| 0.3 | 0.4103 | 0.3274 | 0.2692 |
| 0.4 | 0.5303 | 0.3832 | 0.1294 |
| 0.5 | 0.6355 | 0.3990 | 0.0306 |
| 0.6 | 0.7428 | 0.3965 | 0.0585 |
| 0.7 | 0.8425 | 0.3595 | 0.2010 |
| 0.8 | 0.9183 | 0.2728 | 0.3565 |
| 0.9 | 0.9712 | 0.1489 | 0.5010 |
| 1.0 | 1.0000 | 0.0000 | 0.6324 |

Table 36. Converged control variables, SOGRA, Example 4.2.

| t | u | v |
|---|---|---|
| 0.0 | 0.8972 | -0.6364 |
| 0.1 | 0.8860 | -0.7299 |
| 0.2 | 0.8528 | -0.8071 |
| 0.3 | 0.7982 | -0.8705 |
| 0.4 | 0.7095 | -0.8578 |
| 0.5 | 0.6483 | -0.2684 |
| 0.6 | 0.6926 | 0.6155 |
| 0.7 | 0.5485 | 1.0173 |
| 0.8 | 0.4084 | 0.9488 |
| 0.9 | 0.2583 | 0.8747 |
| 1.0 | 0.1112 | 0.7807 |

$\tau = 1.5830$

<u>Example 4.3.</u> Consider the following problem with bounded state and fixed endpoints:

$$I = \int_0^1 (x^2 + u^2)dt, \tag{115}$$

$$\dot{x} = x^2 - u, \tag{116}$$

$$x - 0.8 - t + t^2 \geq 0, \tag{117}$$

$$x(0) = 1, \tag{118}$$

$$x(1) = 1. \tag{119}$$

By means of the transformation techniques of Ref. 9 , the previous problem can be converted into the following equality constrained problem:

$$I = \int_0^1 (x^2 + u^2)dt, \tag{120}$$

$$\dot{x} = x^2 - u, \qquad \dot{y} = v, \tag{121}$$

$$x^2 - u - 1 + 2t - 2yv = 0, \tag{122}$$

$$x(0) = 1, \qquad y(0) = \sqrt{(0.2)}, \tag{123}$$

$$x(1) = 1. \tag{124}$$

Assume the nominal functions

$$x(t) = 1, \qquad y(t) = \sqrt{(0.2)}, \tag{125}$$

$$u(t) = 1, \qquad v(t) = 1. \tag{126}$$

The results for the sequential conjugate gradient-restoration alogrithm (SCGRA) of Ref. 1 are given in Tables 37-39. The results for the sequential ordinary gradient-restoration algorithm (SOGRA) of Ref. 2 are given in Tables 40-42. Note that SCGRA leads to the solution in $N = 17$ iterations, while SOGRA leads to the solution in $N = 24$ iterations.

Table 37. Convergence history, SCGRA, Example 4.3.

| N | Phase | $\gamma$ | P | Q | I |
|---|---|---|---|---|---|
| 0 | | | 0.21 E+01 | | |
| 1 | REST | | 0.31 E-01 | | |
| 2 | REST | | 0.37 E-03 | | |
| 3 | REST | | 0.83 E-07 | | |
| 4 | REST | | 0.48 E-14 | 0.11 E+01 | 2.43959 |
| 5 | GRAD | $\gamma = 0$ | 0.15 E-01 | | |
| 6 | REST | | 0.54 E-05 | | |
| 7 | REST | | 0.74 E-16 | 0.90 E-02 | 2.10181 |
| 8 | GRAD | $\gamma \neq 0$ | 0.12 E-04 | | |
| 9 | REST | | 0.31 E-11 | 0.15 E-02 | 2.09864 |
| 10 | GRAD | $\gamma \neq 0$ | 0.53 E-04 | | |
| 11 | REST | | 0.25 E-10 | 0.10 E-02 | 2.09530 |
| 12 | GRAD | $\gamma \neq 0$ | 0.99 E-05 | | |
| 13 | REST | | 0.40 E-12 | 0.61 E-03 | 2.09442 |
| 14 | GRAD | $\gamma \neq 0$ | 0.29 E-05 | | |
| 15 | REST | | 0.35 E-13 | 0.17 E-03 | 2.09406 |
| 16 | GRAD | $\gamma \neq 0$ | 0.25 E-05 | | |
| 17 | REST | | 0.20 E-13 | 0.82 E-04 | 2.09377 |

Table 38. Converged state variables, SCGRA, Example 4.3.

| t | x | y |
|---|---|---|
| 0.0 | 1.0000 | 0.4472 |
| 0.1 | 0.9743 | 0.2903 |
| 0.2 | 0.9772 | 0.1312 |
| 0.3 | 1.0101 | 0.0128 |
| 0.4 | 1.0400 | -0.0050 |
| 0.5 | 1.0500 | 0.0012 |
| 0.6 | 1.0400 | -0.0042 |
| 0.7 | 1.0102 | 0.0142 |
| 0.8 | 0.9775 | 0.1324 |
| 0.9 | 0.9748 | 0.2912 |
| 1.0 | 1.0000 | 0.4472 |

Table 39. Converged control variables, SCGRA, Example 4.3.

| t | u | v |
|---|---|---|
| 0.0 | 1.3592 | -1.5196 |
| 0.1 | 1.0589 | -1.5661 |
| 0.2 | 0.7759 | -1.6036 |
| 0.3 | 0.6357 | -0.5975 |
| 0.4 | 0.8823 | 0.0645 |
| 0.5 | 1.1024 | 0.0041 |
| 0.6 | 1.2811 | -0.0555 |
| 0.7 | 1.4035 | 0.5956 |
| 0.8 | 1.1287 | 1.6111 |
| 0.9 | 0.8524 | 1.5413 |
| 1.0 | 0.6142 | 1.5493 |

$\tau = 1.0000$

Table 40. Convergence history, SOGRA, Example 4.3.

| N | Phase | P | Q | I |
|---|---|---|---|---|
| 0 | | 0.21 E+01 | | |
| 1 | REST | 0.31 E-01 | | |
| 2 | REST | 0.37 E-03 | | |
| 3 | REST | 0.83 E-07 | | |
| 4 | REST | 0.48 E-14 | 0.11 E+01 | 2.43959 |
| 5 | GRAD | 0.15 E-01 | | |
| 6 | REST | 0.54 E-05 | | |
| 7 | REST | 0.74 E-16 | 0.90 E-02 | 2.10181 |
| 8 | GRAD | 0.25 E-05 | | |
| 9 | REST | 0.79 E-13 | 0.15 E-02 | 2.09857 |
| 10 | GRAD | 0.39 E-05 | | |
| 11 | REST | 0.23 E-13 | 0.25 E-02 | 2.09694 |
| 12 | GRAD | 0.26 E-06 | | |
| 13 | REST | 0.59 E-15 | 0.62 E-03 | 2.09599 |
| 14 | GRAD | 0.78 E-06 | | |
| 15 | REST | 0.98 E-15 | 0.11 E-02 | 2.09535 |
| 16 | GRAD | 0.66 E-07 | | |
| 17 | REST | 0.33 E-16 | 0.30 E-03 | 2.09492 |
| 18 | GRAD | 0.17 E-06 | | |
| 19 | REST | 0.49 E-16 | 0.59 E-03 | 2.09460 |
| 20 | GRAD | 0.19 E-07 | | |
| 21 | REST | 0.26 E-17 | 0.16 E-03 | 2.09437 |
| 22 | GRAD | 0.51 E-07 | | |
| 23 | REST | 0.42 E-17 | 0.33 E-03 | 2.09420 |
| 24 | GRAD | 0.64 E-08 | 0.97 E-04 | 2.09397 |

Table 41. Converged state variables, SOGRA, Example 4.3.

| t | x | y |
|---|---|---|
| 0.0 | 1.0000 | 0.4472 |
| 0.1 | 0.9748 | 0.2912 |
| 0.2 | 0.9786 | 0.1367 |
| 0.3 | 1.0104 | 0.0223 |
| 0.4 | 1.0400 | -0.0064 |
| 0.5 | 1.0500 | -0.0070 |
| 0.6 | 1.0400 | -0.0065 |
| 0.7 | 1.0104 | 0.0224 |
| 0.8 | 0.9786 | 0.1367 |
| 0.9 | 0.9748 | 0.2912 |
| 1.0 | 1.0000 | 0.4472 |

Table 42. Converged control variables, SOGRA, Example 4.3.

| t | u | v |
|---|---|---|
| 0.0 | 1.3993 | -1.5644 |
| 0.1 | 1.0571 | -1.5565 |
| 0.2 | 0.7664 | -1.4938 |
| 0.3 | 0.6505 | -0.6596 |
| 0.4 | 0.8809 | -0.0529 |
| 0.5 | 1.1025 | -0.0008 |
| 0.6 | 1.2823 | 0.0541 |
| 0.7 | 1.3912 | 0.6613 |
| 0.8 | 1.1495 | 1.4921 |
| 0.9 | 0.8434 | 1.5566 |
| 1.0 | 0.6007 | 1.5644 |

$\tau = 1.0000$

Example 4.4. Consider the following problem with bounded state and fixed endpoints:

$$I = \tau \int_0^1 (u^2 - x^2 + \tau t)dt \,, \tag{127}$$

$$\dot{x} = \tau u, \qquad \dot{y} = \tau(2 - 4x^2), \tag{128}$$

$$1 - y \geq 0, \tag{129}$$

$$x(0) = 0, \qquad y(0) = 0, \tag{130}$$

$$x(1) = 1, \qquad y(1) = 0, \tag{131}$$

where $\tau = \pi/2$. By means of the transformation techniques of Ref. 9, the previous problem can be converted into the following equality constrained problem:

$$I = \tau \int_0^1 (u^2 - x^2 + \tau t)dt, \tag{132}$$

$$\dot{x} = \tau u, \qquad \dot{y} = \tau(2 - 4x^2), \qquad \dot{z} = \tau h, \qquad \dot{h} = \tau v, \tag{133}$$

$$4xu - zv - h^2 = 0, \tag{134}$$

$$x(0) = 0, \qquad y(0) = 0, \qquad z(0) = 1, \qquad h(0) = -1, \tag{135}$$

$$x(1) = 1, \qquad y(1) = 0. \tag{136}$$

Assume the nominal functions

$$x(t) = t, \quad y(t) = 4t(1-t), \quad z(t) = 1-2t, \quad h(t) = -1, \tag{137}$$

$$u(t) = 1/\tau, \quad v(t) = 0. \tag{138}$$

The results for the sequential conjugate gradient-restoration algorithm (SCGRA) of Ref. 1 are given in Tables 43-45. The results for the sequential ordinary gradient-restoration algorithm (SOGRA) of Ref. 2 are given in Tables 46-48. Note that SCGRA leads to the solution in $N = 16$ iterations, while SOGRA leads to the solution in $N = 21$ iterations.

Table 43. Convergence history, SCGRA, Example 4.4.

| N | Phase | $\gamma$ | P | Q | I |
|---|---|---|---|---|---|
| 0 | | | 0.23 E+01 | | |
| 1 | REST | | 0.94 E-01 | | |
| 2 | REST | | 0.13 E-03 | | |
| 3 | REST | | 0.85 E-08 | 0.18 E-01 | 1.24356 |
| 4 | GRAD | $\gamma = 0$ | 0.94 E-05 | | |
| 5 | REST | | 0.19 E-11 | 0.55 E-02 | 1.23748 |
| 6 | GRAD | $\gamma \neq 0$ | 0.49 E-05 | | |
| 7 | REST | | 0.58 E-12 | 0.21 E-02 | 1.23547 |
| 8 | GRAD | $\gamma \neq 0$ | 0.12 E-05 | | |
| 9 | REST | | 0.37 E-13 | 0.98 E-03 | 1.23457 |
| 10 | GRAD | $\gamma \neq 0$ | 0.38 E-06 | | |
| 11 | REST | | 0.26 E-14 | 0.51 E-03 | 1.23412 |
| 12 | GRAD | $\gamma \neq 0$ | 0.12 E-06 | | |
| 13 | REST | | 0.18 E-15 | 0.24 E-03 | 1.23391 |
| 14 | GRAD | $\gamma \neq 0$ | 0.34 E-07 | | |
| 15 | REST | | 0.11 E-16 | 0.11 E-03 | 1.23380 |
| 16 | GRAD | $\gamma \neq 0$ | 0.86 E-08 | 0.53 E-04 | 1.23375 |

Table 44. Converged state variables, SCGRA, Example 4.4.

| t | x | y | z | h |
|---|---|---|---|---|
| 0.0 | 0.0000 | 0.0000 | 1.0000 | -1.0000 |
| 0.1 | 0.1564 | 0.3090 | 0.8312 | -1.1441 |
| 0.2 | 0.3088 | 0.5878 | 0.6419 | -1.2604 |
| 0.3 | 0.4539 | 0.8093 | 0.4366 | -1.3461 |
| 0.4 | 0.5883 | 0.9509 | 0.2214 | -1.3892 |
| 0.5 | 0.7063 | 1.0000 | 0.0014 | -1.4071 |
| 0.6 | 0.8081 | 0.9519 | -0.2191 | -1.3968 |
| 0.7 | 0.8910 | 0.8104 | -0.4354 | -1.3501 |
| 0.8 | 0.9517 | 0.5887 | -0.6413 | -1.2653 |
| 0.9 | 0.9880 | 0.3092 | -0.8311 | -1.1462 |
| 1.0 | 1.0000 | 0.0000 | -0.9999 | -1.0000 |

Table 45. Converged control variables, SCGRA, Example 4.4.

| t | u | v |
|---|---|---|
| 0.0 | 0.9978 | -1.0000 |
| 0.1 | 0.9867 | -0.8320 |
| 0.2 | 0.9424 | -0.6609 |
| 0.3 | 0.9038 | -0.3907 |
| 0.4 | 0.8024 | -0.1867 |
| 0.5 | 0.7008 | -0.0349 |
| 0.6 | 0.5917 | 0.1759 |
| 0.7 | 0.4601 | 0.4200 |
| 0.8 | 0.3101 | 0.6555 |
| 0.9 | 0.1528 | 0.8535 |
| 1.0 | 0.0003 | 0.9986 |

$\tau = 1.5707$

Table 46. Convergence history, SOGRA, Example 4.4.

| N | Phase | P | Q | I |
|---|---|---|---|---|
| 0 | | 0.23 E+01 | | |
| 1 | REST | 0.94 E-01 | | |
| 2 | REST | 0.13 E-03 | | |
| 3 | REST | 0.85 E-08 | 0.18 E-01 | 1.24356 |
| 4 | GRAD | 0.94 E-05 | | |
| 5 | REST | 0.19 E-11 | 0.55 E-02 | 1.23748 |
| 6 | GRAD | 0.18 E-06 | | |
| 7 | REST | 0.56 E-16 | 0.20 E-02 | 1.23594 |
| 8 | GRAD | 0.71 E-07 | | |
| 9 | REST | 0.74 E-17 | 0.15 E-02 | 1.23520 |
| 10 | GRAD | 0.14 E-07 | | |
| 11 | REST | 0.15 E-18 | 0.75 E-03 | 1.23477 |
| 12 | GRAD | 0.76 E-08 | 0.70 E-03 | 1.23449 |
| 13 | GRAD | 0.19 E-07 | | |
| 14 | REST | 0.12 E-18 | 0.38 E-03 | 1.23430 |
| 15 | GRAD | 0.15 E-08 | 0.35 E-03 | 1.23416 |
| 16 | GRAD | 0.43 E-08 | 0.22 E-03 | 1.23406 |
| 17 | GRAD | 0.73 E-08 | 0.18 E-03 | 1.23399 |
| 18 | GRAD | 0.10 E-07 | | |
| 19 | REST | 0.21 E-19 | 0.14 E-03 | 1.23393 |
| 20 | GRAD | 0.14 E-09 | 0.10 E-03 | 1.23389 |
| 21 | GRAD | 0.44 E-09 | 0.98 E-04 | 1.23385 |

Table 47. Converged state variables, SOGRA, Example 4.4.

| t | x | y | z | h |
|---|---|---|---|---|
| 0.0 | 0.0000 | 0.0000 | 1.0000 | -1.0000 |
| 0.1 | 0.1568 | 0.3089 | 0.8312 | -1.1437 |
| 0.2 | 0.3097 | 0.5875 | 0.6421 | -1.2583 |
| 0.3 | 0.4553 | 0.8082 | 0.4378 | -1.3367 |
| 0.4 | 0.5876 | 0.9497 | 0.2240 | -1.3809 |
| 0.5 | 0.7045 | 0.9999 | 0.0051 | -1.4029 |
| 0.6 | 0.8064 | 0.9536 | -0.2152 | -1.3974 |
| 0.7 | 0.8904 | 0.8133 | -0.4319 | -1.3558 |
| 0.8 | 0.9525 | 0.5915 | -0.6390 | -1.2745 |
| 0.9 | 0.9896 | 0.3104 | -0.8303 | -1.1545 |
| 1.0 | 1.0000 | 0.0000 | -1.0000 | -0.9999 |

Table 48. Converged control variables, SOGRA, Example 4.1.

| t | u | v |
|---|---|---|
| 0.0 | 1.0215 | -1.0000 |
| 0.1 | 0.9889 | -0.8274 |
| 0.2 | 0.9541 | -0.6249 |
| 0.3 | 0.8915 | -0.3719 |
| 0.4 | 0.7914 | -0.2083 |
| 0.5 | 0.6982 | -0.0632 |
| 0.6 | 0.5957 | 0.1432 |
| 0.7 | 0.4688 | 0.3899 |
| 0.8 | 0.3184 | 0.6435 |
| 0.9 | 0.1522 | 0.8797 |
| 1.0 | -0.0203 | 1.0814 |

$\tau = 1.5707$

Example 4.5. Consider the following problem with bounded state and fixed endpoints:

$$I = \int_0^1 u^2 dt, \tag{139}$$

$$\dot{x} = y, \qquad \dot{y} = u, \tag{140}$$

$$0.15 - x \geq 0, \tag{141}$$

$$x(0) = 0, \qquad y(0) = 1, \tag{142}$$

$$x(1) = 0, \qquad y(1) = -1. \tag{143}$$

By means of the transformation techniques of Ref. 9, the previous problem can be converted into the following equality constrained problem:

$$I = \int_0^1 u^2 dt, \tag{144}$$

$$\dot{x} = y, \quad \dot{y} = u, \quad \dot{z} = h, \quad \dot{h} = v, \tag{145}$$

$$u + 2zv + 2h^2 = 0, \tag{146}$$

$$x(0) = 0, \quad y(0) = 1, \quad z(0) = \sqrt{(0.15)}, \quad h(0) = -1/2\sqrt{(0.15)}, \tag{147}$$

$$x(1) = 0, \quad y(1) = -1. \tag{148}$$

Assume the nominal functions

$$x(t)=0, \quad y(t)=1-2t, \quad z(t)=\surd(0.15)(1-2t), \quad h(t)=(2t-1)/2\surd(0.15), \tag{149}$$

$$u(t)=1, \quad v(t)=0. \tag{150}$$

The results for the sequential conjugate gradient-restoration algorithm (SCGRA) of Ref. 1 are given in Tables 49-51. The results for the sequential ordinary gradient-restoration algorithm (SOGRA) of Ref. 2 are given in Tables 52-54. Note that SCGRA leads to the solution in $N=15$ iterations, while SOGRA leads to the solution in $N=16$ iterations.

Table 49. Convergence history, SCGRA, Example 4.5.

| N | Phase | $\gamma$ | P | Q | I |
|---|---|---|---|---|---|
| 0 | | | 0.22 E+02 | | |
| 1 | REST | | 0.46 E+01 | | |
| 2 | REST | | 0.11 E+00 | | |
| 3 | REST | | 0.22 E-02 | | |
| 4 | REST | | 0.11 E-05 | | |
| 5 | REST | | 0.44 E-13 | 0.11 E+00 | 6.03009 |
| 6 | GRAD | $\gamma = 0$ | 0.81 E-04 | | |
| 7 | REST | | 0.15 E-14 | 0.79 E-02 | 5.93793 |
| 8 | GRAD | $\gamma \neq 0$ | 0.28 E-05 | | |
| 9 | REST | | 0.57 E-16 | 0.14 E-02 | 5.92940 |
| 10 | GRAD | $\gamma \neq 0$ | 0.53 E-06 | | |
| 11 | REST | | 0.22 E-17 | 0.41 E-03 | 5.92743 |
| 12 | GRAD | $\gamma \neq 0$ | 0.15 E-06 | | |
| 13 | REST | | 0.12 E-18 | 0.15 E-03 | 5.92677 |
| 14 | GRAD | $\gamma \neq 0$ | 0.50 E-07 | | |
| 15 | REST | | 0.93 E-20 | 0.65 E-04 | 5.92650 |

Table 50. Converged state variables, SCGRA, Example 4.5.

| t | x | y | z | h |
|---|---|---|---|---|
| 0.0 | 0.0000 | 1.0000 | 0.3872 | -1.2909 |
| 0.1 | 0.0794 | 0.6047 | 0.2656 | -1.1381 |
| 0.2 | 0.1242 | 0.3088 | 0.1603 | -0.9629 |
| 0.3 | 0.1443 | 0.1100 | 0.0748 | -0.7354 |
| 0.4 | 0.1497 | 0.0136 | 0.0164 | -0.4141 |
| 0.5 | 0.1499 | -0.0001 | -0.0051 | -0.0109 |
| 0.6 | 0.1497 | -0.0121 | 0.0148 | 0.4089 |
| 0.7 | 0.1445 | -0.1110 | 0.0740 | 0.7502 |
| 0.8 | 0.1243 | -0.3085 | 0.1602 | 0.9625 |
| 0.9 | 0.0794 | -0.6054 | 0.2656 | 1.1395 |
| 1.0 | 0.0000 | -1.0000 | 0.3872 | 1.2909 |

Table 51. Converged control variables, SCGRA, Example 4.5.

| t | u | v |
|---|---|---|
| 0.0 | -4.4555 | 1.4487 |
| 0.1 | -3.4684 | 1.6515 |
| 0.2 | -2.4732 | 1.9287 |
| 0.3 | -1.4837 | 2.6862 |
| 0.4 | -0.4657 | 3.7230 |
| 0.5 | 0.0435 | 4.2231 |
| 0.6 | -0.4541 | 4.0244 |
| 0.7 | -1.5173 | 2.6445 |
| 0.8 | -2.4554 | 1.8790 |
| 0.9 | -3.4483 | 1.6020 |
| 1.0 | -4.4054 | 1.3840 |

$\tau = 1.0000$

Table 52. Convergence history, SOGRA, Example 4.5.

| N | Phase | P | Q | I |
|---|---|---|---|---|
| 0 | | 0.22 E+02 | | |
| 1 | REST | 0.46 E+01 | | |
| 2 | REST | 0.11 E+00 | | |
| 3 | REST | 0.22 E-02 | | |
| 4 | REST | 0.11 E-05 | | |
| 5 | REST | 0.44 E-13 | 0.11 E+00 | 6.03009 |
| 6 | GRAD | 0.81 E-04 | | |
| 7 | REST | 0.15 E-14 | 0.79 E-02 | 5.93793 |
| 8 | GRAD | 0.11 E-05 | | |
| 9 | REST | 0.28 E-17 | 0.20 E-02 | 5.93016 |
| 10 | GRAD | 0.16 E-06 | | |
| 11 | REST | 0.12 E-13 | 0.74 E-03 | 5.92817 |
| 12 | GRAD | 0.24 E-07 | | |
| 13 | REST | 0.15 E-20 | 0.37 E-03 | 5.92738 |
| 14 | GRAD | 0.86 E-08 | 0.20 E-03 | 5.92687 |
| 15 | GRAD | 0.62 E-08 | 0.12 E-03 | 5.92661 |
| 16 | GRAD | 0.74 E-08 | 0.52 E-04 | 5.92650 |

Table 53. Converged state variables, SOGRA, Example 4.5.

| t | x | y | z | h |
|---|---|---|---|---|
| 0.0 | 0.0000 | 1.0000 | 0.3872 | -1.2909 |
| 0.1 | 0.0793 | 0.6045 | 0.2657 | -1.1375 |
| 0.2 | 0.1242 | 0.3078 | 0.1606 | -0.9583 |
| 0.3 | 0.1442 | 0.1105 | 0.0756 | -0.7301 |
| 0.4 | 0.1496 | 0.0145 | 0.0175 | -0.4152 |
| 0.5 | 0.1499 | -0.0001 | -0.0045 | -0.0174 |
| 0.6 | 0.1497 | -0.0118 | 0.0147 | 0.4010 |
| 0.7 | 0.1446 | -0.1097 | 0.0733 | 0.7483 |
| 0.8 | 0.1244 | -0.3098 | 0.1599 | 0.9688 |
| 0.9 | 0.0794 | -0.6057 | 0.2655 | 1.1403 |
| 1.0 | 0.0000 | -1.0000 | 0.3872 | 1.2909 |

Table 54. Converged control variables, SOGRA, Example 4.5.

| t | u | v |
|---|---|---|
| 0.0 | -4.4535 | 1.4461 |
| 0.1 | -3.4592 | 1.6392 |
| 0.2 | -2.4746 | 1.9862 |
| 0.3 | -1.4669 | 2.6475 |
| 0.4 | -0.4730 | 3.6484 |
| 0.5 | 0.0375 | 4.1943 |
| 0.6 | -0.4405 | 4.0278 |
| 0.7 | -1.5262 | 2.7693 |
| 0.8 | -2.4681 | 1.8476 |
| 0.9 | -3.4508 | 1.6007 |
| 1.0 | -4.4354 | 1.4228 |

$\tau = 1.0000$

Example 4.6. Consider the following problem with bounded state and free final time:

$$I = \tau , \tag{151}$$

$$\dot{x} = \tau u, \qquad \dot{y} = \tau v, \tag{152}$$

$$u^2 + v^2 - 1 = 0, \qquad (x - 2)^2 + y^2 - 1 \geq 0, \tag{153}$$

$$x(0) = 0, \qquad y(0) = 0, \tag{154}$$

$$x(1) = 4, \qquad y(1) = 0. \tag{155}$$

By means of the transformation techniques of Ref. 9 , the previous problem can be converted into the following equality constrained problem:

$$I = \tau, \tag{156}$$

$$\dot{x} = \tau u, \qquad \dot{y} = \tau v, \qquad \dot{z} = \tau w, \tag{157}$$

$$u^2 + v^2 - 1 = 0, \qquad (x-2)u + yv - zw = 0, \tag{158}$$

$$x(0) = 0, \qquad y(0) = 0, \qquad z(0) = \sqrt{3}, \tag{159}$$

$$x(1) = 4, \qquad y(1) = 0. \tag{160}$$

Assume the nominal functions

$$x(t) = 4t, \qquad y(t) = 4t(1-t), \qquad z(t) = \sqrt{3}, \tag{161}$$

$$u(t) = 1, \qquad v(t) = 1-2t, \qquad w(t) = 0, \tag{162}$$

$$\tau = 4. \tag{163}$$

The results for the sequential conjugate gradient-restoration algorithm (SCGRA) of Ref. 1 are given in Tables 55-57. The results for the sequential ordinary gradient-restoration algorithm (SOGRA) of Ref. 2 are given in Tables 58-60. Note that SCGRA leads to the solution in $N = 20$ iterations, while SOGRA leads to the solution in $N = 24$ iterations.

Table 55. Convergence history, SCGRA, Example 4.6.

| N | Phase | $\gamma$ | P | Q | I |
|---|---|---|---|---|---|
| 0 | | | 0.10 E+01 | | |
| 1 | REST | | 0.76 E-01 | | |
| 2 | REST | | 0.53 E-02 | | |
| 3 | REST | | 0.10 E-03 | | |
| 4 | REST | | 0.12 E-07 | | |
| 5 | REST | | 0.14 E-15 | 0.95 E-01 | 4.52789 |
| 6 | GRAD | $\gamma = 0$ | 0.99 E-04 | | |
| 7 | REST | | 0.24 E-07 | | |
| 8 | REST | | 0.31 E-16 | 0.93 E-02 | 4.51433 |
| 9 | GRAD | $\gamma \neq 0$ | 0.54 E-05 | | |
| 10 | REST | | 0.11 E-09 | 0.82 E-03 | 4.51288 |
| 11 | GRAD | $\gamma \neq 0$ | 0.10 E-05 | | |
| 12 | REST | | 0.41 E-11 | 0.88 E-03 | 4.51241 |
| 13 | GRAD | $\gamma \neq 0$ | 0.19 E-05 | | |
| 14 | REST | | 0.13 E-10 | 0.93 E-03 | 4.51198 |
| 15 | GRAD | $\gamma \neq 0$ | 0.31 E-05 | | |
| 16 | REST | | 0.31 E-10 | 0.55 E-03 | 4.51164 |
| 17 | GRAD | $\gamma \neq 0$ | 0.92 E-06 | | |
| 18 | REST | | 0.29 E-11 | 0.13 E-03 | 4.51150 |
| 19 | GRAD | $\gamma \neq 0$ | 0.18 E-06 | | |
| 20 | REST | | 0.12 E-12 | 0.67 E-04 | 4.51144 |

Table 56. Converged state variables, SCGRA, Example 4.6.

| t | x | y | z |
|---|---|---|---|
| 0.0 | 0.0000 | 0.0000 | 1.7320 |
| 0.1 | 0.3911 | 0.2247 | 1.2801 |
| 0.2 | 0.7819 | 0.4501 | 0.8284 |
| 0.3 | 1.1730 | 0.6751 | 0.3737 |
| 0.4 | 1.5639 | 0.9000 | 0.0131 |
| 0.5 | 1.9999 | 1.0000 | 0.0010 |
| 0.6 | 2.4360 | 0.9000 | 0.0134 |
| 0.7 | 2.8270 | 0.6751 | 0.3738 |
| 0.8 | 3.2180 | 0.4502 | 0.8284 |
| 0.9 | 3.6088 | 0.2247 | 1.2801 |
| 1.0 | 4.0000 | 0.0000 | 1.7320 |

Table 57. Converged control variables, SCGRA, Example 4.6.

| t | u | v | w |
|---|---|---|---|
| 0.0 | 0.8648 | 0.5019 | -0.9986 |
| 0.1 | 0.8667 | 0.4987 | -1.0017 |
| 0.2 | 0.8683 | 0.4958 | -1.0073 |
| 0.3 | 0.8685 | 0.4956 | -1.0264 |
| 0.4 | 0.9015 | 0.4326 | -0.2789 |
| 0.5 | 1.0000 | 0.0000 | 0.0004 |
| 0.6 | 0.9015 | -0.4326 | 0.2791 |
| 0.7 | 0.8683 | -0.4959 | 1.0251 |
| 0.8 | 0.8684 | -0.4958 | 1.0073 |
| 0.9 | 0.8668 | -0.4985 | 1.0018 |
| 1.0 | 0.8646 | -0.5023 | 0.9983 |

$\tau = 4.5114$

Table 58. Convergence history, SOGRA, Example 4.6.

| N | Phase | P | Q | I |
|---|---|---|---|---|
| 0 | | 0.10 E+01 | | |
| 1 | REST | 0.76 E-01 | | |
| 2 | REST | 0.53 E-02 | | |
| 3 | REST | 0.10 E-03 | | |
| 4 | REST | 0.12 E-07 | | |
| 5 | REST | 0.14 E-15 | 0.95 E-01 | 4.52789 |
| 6 | GRAD | 0.99 E-04 | | |
| 7 | REST | 0.24 E-07 | | |
| 8 | REST | 0.31 E-16 | 0.93 E-02 | 4.51433 |
| 9 | GRAD | 0.40 E-06 | | |
| 10 | REST | 0.38 E-12 | 0.13 E-02 | 4.51308 |
| 11 | GRAD | 0.10 E-06 | | |
| 12 | REST | 0.36 E-13 | 0.12 E-02 | 4.51269 |
| 13 | GRAD | 0.14 E-07 | | |
| 14 | REST | 0.91 E-15 | 0.52 E-03 | 4.51249 |
| 15 | GRAD | 0.25 E-07 | | |
| 16 | REST | 0.18 E-14 | 0.79 E-03 | 4.51233 |
| 17 | GRAD | 0.64 E-08 | 0.35 E-03 | 4.51208 |
| 18 | GRAD | 0.57 E-08 | 0.23 E-03 | 4.51196 |
| 19 | GRAD | 0.15 E-06 | | |
| 20 | REST | 0.15 E-12 | 0.11 E-02 | 4.51190 |
| 21 | GRAD | 0.50 E-08 | 0.11 E-03 | 4.51161 |
| 22 | GRAD | 0.59 E-08 | 0.16 E-03 | 4.51152 |
| 23 | GRAD | 0.90 E-08 | 0.13 E-03 | 4.51145 |
| 24 | GRAD | 0.94 E-08 | 0.65 E-04 | 4.51142 |

Table 59. Converged state variables, SOGRA, Example 4.6.

| t | x | y | z |
|---|---|---|---|
| 0.0 | 0.0000 | 0.0000 | 1.7320 |
| 0.1 | 0.3905 | 0.2258 | 1.2810 |
| 0.2 | 0.7811 | 0.4516 | 0.8303 |
| 0.3 | 1.1716 | 0.6774 | 0.3810 |
| 0.4 | 1.5638 | 0.9002 | 0.0309 |
| 0.5 | 1.9999 | 1.0000 | -0.0182 |
| 0.6 | 2.4361 | 0.9002 | 0.0309 |
| 0.7 | 2 8283 | 0.6774 | 0.3810 |
| 0.8 | 3.2188 | 0.4516 | 0.8303 |
| 0.9 | 3.6094 | 0.2258 | 1.2810 |
| 1.0 | 4.0000 | 0.0000 | 1.7320 |

Table 60. Converged control variables, SOGRA, Example 4.6.

| t | u | v | w |
|---|---|---|---|
| 0.0 | 0.8657 | 0.5005 | -0.9997 |
| 0.1 | 0.8657 | 0.5005 | -0.9994 |
| 0.2 | 0.8657 | 0.5004 | -0.9984 |
| 0.3 | 0.8652 | 0.5013 | -0.9892 |
| 0.4 | 0.9048 | 0.4258 | -0.3639 |
| 0.5 | 1.0000 | 0.0000 | 0.0000 |
| 0.6 | 0.9048 | -0.4258 | 0.3636 |
| 0.7 | 0.8652 | -0.5013 | 0.9892 |
| 0.8 | 0.8657 | -0.5004 | 0.9984 |
| 0.9 | 0.8657 | -0.5005 | 0.9994 |
| 1.0 | 0.8657 | -0.5005 | 0.9997 |

$\tau = 4.5114$

Example 4.7. Consider the following problem with bounded state and free end coordinates:

$$I = \int_0^1 (x^2 + y^2 + u^2/200)dt, \tag{164}$$

$$\dot{x} = y, \qquad \dot{y} = u - y, \tag{165}$$

$$8(t-1/2)^2 - y - 1/2 \geq 0 , \tag{166}$$

$$x(0) = 0, \qquad y(0) = -1. \tag{167}$$

By means of the transformation techniques of Ref. 9, the previous problem can be converted into the following equality constrained problem:

$$I = \int_0^1 (x^2 + y^2 + u^2/200)dt, \tag{168}$$

$$\dot{x} = y, \qquad \dot{y} = u - y, \qquad \dot{z} = v, \tag{169}$$

$$16(t-1/2) - u + y - 2zv = 0, \tag{170}$$

$$x(0) = 0, \qquad y(0) = -1, \qquad z(0) = \sqrt{(5/2)}. \tag{171}$$

Assume the nominal functions

$$x(t) = t, \qquad y(t) = t-1, \qquad z(t) = [1 - \sqrt{(5/2)}]t + \sqrt{(5/2)}, \tag{172}$$

$$u(t) = 1, \qquad v(t) = 1. \tag{173}$$

The results for the sequential conjugate gradient-restoration algorithm (SCGRA) of Ref. 1 are given in Tables 61-63. The results for the sequential ordinary gradient-restoration algorithm (SOGRA) of Ref. 2 are given in Tables 64-66. Note that SCGRA leads to the solution in $N = 23$ iterations, while SOGRA leads to the solution in $N = 19$ iterations.

Table 61. Convergence history, SCGRA, Example 4.7.

| N | Phase | $\gamma$ | P | Q | I |
|---|---|---|---|---|---|
| 0 | | | 0.49 E+02 | | |
| 1 | REST | | 0.14 E+01 | | |
| 2 | REST | | 0.61 E-01 | | |
| 3 | REST | | 0.81 E-04 | | |
| 4 | REST | | 0.10 E-10 | 0.20 E+00 | 0.61395 |
| 5 | GRAD | $\gamma = 0$ | 0.12 E+00 | | |
| 6 | REST | | 0.14 E-04 | | |
| 7 | REST | | 0.20 E-11 | 0.22 E-01 | 0.33368 |
| 8 | GRAD | $\gamma \neq 0$ | 0.47 E-01 | | |
| 9 | REST | | 0.49 E-05 | | |
| 10 | REST | | 0.11 E-13 | 0.66 E-02 | 0.25889 |
| 11 | GRAD | $\gamma \neq 0$ | 0.41 E-01 | | |
| 12 | REST | | 0.13 E-05 | | |
| 13 | REST | | 0.16 E-14 | 0.29 E-02 | 0.21259 |
| 14 | GRAD | $\gamma \neq 0$ | 0.29 E-01 | | |
| 15 | REST | | 0.10 E-06 | | |
| 16 | REST | | 0.52 E-17 | 0.10 E-02 | 0.18573 |
| 17 | GRAD | $\gamma \neq 0$ | 0.67 E-02 | | |
| 18 | REST | | 0.23 E-07 | | |
| 19 | REST | | 0.39 E-18 | 0.51 E-03 | 0.17768 |
| 20 | GRAD | $\gamma \neq 0$ | 0.45 E-02 | | |
| 21 | REST | | 0.86 E-08 | 0.30 E-03 | 0.17343 |
| 22 | GRAD | $\gamma \neq 0$ | 0.65 E-03 | | |
| 23 | REST | | 0.39 E-10 | 0.45 E-04 | 0.17177 |

Table 62. Converged state variables, SCGRA, Example 4.7.

| t | x | y | z |
|---|---|---|---|
| 0.0 | 0.0000 | -1.0000 | 1.5811 |
| 0.1 | -0.0482 | -0.1315 | 0.9547 |
| 0.2 | -0.0494 | 0.0257 | 0.4407 |
| 0.3 | -0.0555 | -0.1856 | 0.0749 |
| 0.4 | -0.0870 | -0.4208 | -0.0283 |
| 0.5 | -0.1344 | -0.5003 | -0.0197 |
| 0.6 | -0.1817 | -0.4200 | 0.0082 |
| 0.7 | -0.2137 | -0.2067 | 0.1635 |
| 0.8 | -0.2247 | -0.0369 | 0.5068 |
| 0.9 | -0.2255 | 0.0087 | 0.8781 |
| 1.0 | -0.2234 | 0.0410 | 1.2078 |

Table 63. Converged control variables, SCGRA, Example 4.7.

| t | u | v |
|---|---|---|
| 0.0 | 13.3645 | -7.0723 |
| 0.1 | 4.1741 | -5.6064 |
| 0.2 | -0.6737 | -4.6516 |
| 0.3 | -3.0397 | -2.3064 |
| 0.4 | -2.0288 | -0.1417 |
| 0.5 | -0.4944 | 0.1512 |
| 0.6 | 1.1697 | 0.6174 |
| 0.7 | 2.1149 | 2.6857 |
| 0.8 | 0.9142 | 3.7965 |
| 0.9 | 0.1801 | 3.5463 |
| 1.0 | 0.7636 | 3.0124 |

$\tau = 1.0000$

Table 64. Convergence history, SOGRA, Example 4.7.

| N | Phase | P | Q | I |
|---|---|---|---|---|
| 0 | | 0.49 E+02 | | |
| 1 | REST | 0.14 E+01 | | |
| 2 | REST | 0.61 E-01 | | |
| 3 | REST | 0.81 E-04 | | |
| 4 | REST | 0.10 E-10 | 0.20 E+00 | 0.61395 |
| 5 | GRAD | 0.12 E+00 | | |
| 6 | REST | 0.14 E-04 | | |
| 7 | REST | 0.20 E-11 | 0.22 E-01 | 0.33368 |
| 8 | GRAD | 0.35 E-01 | | |
| 9 | REST | 0.70 E-06 | | |
| 10 | REST | 0.46 E-15 | 0.46 E-02 | 0.23716 |
| 11 | GRAD | 0.20 E-01 | | |
| 12 | REST | 0.21 E-05 | | |
| 13 | REST | 0.14 E-14 | 0.13 E-02 | 0.18457 |
| 14 | GRAD | 0.32 E-03 | | |
| 15 | REST | 0.10 E-09 | 0.38 E-03 | 0.17778 |
| 16 | GRAD | 0.12 E-02 | | |
| 17 | REST | 0.22 E-08 | 0.42 E-03 | 0.17460 |
| 18 | GRAD | 0.31 E-04 | | |
| 19 | REST | 0.16 E-11 | 0.83 E-04 | 0.17294 |

Table 65. Converged state variables, SOGRA, Example 4.7.

| t | x | y | z |
|---|---|---|---|
| 0.0 | 0.0000 | -1.0000 | 1.5811 |
| 0.1 | -0.0529 | -0.2008 | 0.9903 |
| 0.2 | -0.0604 | -0.0230 | 0.4929 |
| 0.3 | -0.0691 | -0.1922 | 0.1107 |
| 0.4 | -0.1007 | -0.4212 | -0.0353 |
| 0.5 | -0.1482 | -0.5026 | -0.0519 |
| 0.6 | -0.1958 | -0.4203 | -0.0188 |
| 0.7 | -0.2276 | -0.2051 | 0.1585 |
| 0.8 | -0.2386 | -0.0370 | 0.5070 |
| 0.9 | -0.2390 | 0.0168 | 0.8736 |
| 1.0 | -0.2354 | 0.0643 | 1.1981 |

Table 66. Converged control variables, SOGRA, Example 4.7.

| t | u | v |
|---|---|---|
| 0.0 | 11.8420 | -6.5908 |
| 0.1 | 4.0067 | -5.3553 |
| 0.2 | -0.2976 | -4.5898 |
| 0.3 | -2.7911 | -2.7144 |
| 0.4 | -2.0564 | -0.4986 |
| 0.5 | -0.4991 | 0.0336 |
| 0.6 | 1.2103 | 0.8156 |
| 0.7 | 2.0920 | 2.8465 |
| 0.8 | 0.9387 | 3.7712 |
| 0.9 | 0.3198 | 3.4895 |
| 1.0 | 0.9359 | 2.9747 |

$\tau = 1.0000$

Example 4.8. Consider the following problem with bounded state and free end coordinates:

$$I = \int_0^1 (x^2 + y^2 + u^2/200)dt, \tag{174}$$

$$\dot{x} = y, \qquad \dot{y} = u-y, \tag{175}$$

$$8(t-1/2)^2 - x - 1/2 \geq 0, \tag{176}$$

$$x(0) = 0, \qquad y(0) = -1. \tag{177}$$

By means of the transformation techniques of Ref. 9 , the previous problem can be converted into the following equality constrained problem:

$$I = \int_0^1 (x^2 + y^2 + u^2/200)dt, \tag{178}$$

$$\dot{x} = y, \qquad \dot{y} = u-y, \qquad \dot{z} = h, \qquad \dot{h} = v, \tag{179}$$

$$16 - u + y - 2h^2 - 2zv = 0, \tag{180}$$

$$x(0) = 0, \qquad y(0) = -1, \qquad z(0) = \sqrt{(3/2)}, \quad h(0) = -7/\sqrt{6}. \tag{181}$$

Assume the nominal functions

$$x(t) = -t, \qquad y(t) = t-1, \qquad z(t) = \sqrt{(3/2)}, \quad h(t) = -7/\sqrt{6}, \tag{182}$$

$$u(t) = 1, \qquad v(t) = 1. \tag{183}$$

The results for the sequential conjugate gradient-restoration algorithm (SCGRA) of Ref. 1 are given in Tables 67-69. The results for the sequential ordinary gradient-restoration algorithm (SOGRA) of Ref. 2 are given in Tables 70-72. Note that SCGRA leads to the solution in $N = 12$ iterations, while SOGRA leads to the solution in $N = 22$ iterations.

Table 67. Convergence history, SCGRA, Example 4.8.

| N | Phase | $\gamma$ | P | Q | I |
|---|---|---|---|---|---|
| 0 | | | 0.28 E+02 | | |
| 1 | REST | | 0.87 E+01 | | |
| 2 | REST | | 0.59 E-01 | | |
| 3 | REST | | 0.19 E-05 | | |
| 4 | REST | | 0.30 E-15 | 0.85 E-01 | 0.97910 |
| 5 | GRAD | $\gamma = 0$ | 0.17 E-02 | | |
| 6 | REST | | 0.19 E-08 | 0.32 E-02 | 0.79284 |
| 7 | GRAD | $\gamma \neq 0$ | 0.30 E-02 | | |
| 8 | REST | | 0.29 E-08 | 0.10 E-02 | 0.75474 |
| 9 | GRAD | $\gamma \neq 0$ | 0.84 E-04 | | |
| 10 | REST | | 0.15 E-11 | 0.43 E-03 | 0.75113 |
| 11 | GRAD | $\gamma \neq 0$ | 0.60 E-03 | | |
| 12 | REST | | 0.67 E-11 | 0.66 E-04 | 0.74267 |

Table 68. Converged state variables, SCGRA, Example 4.8.

| t | x | y | z | h |
|---|---|---|---|---|
| 0.0 | 0.0000 | -1.0000 | 1.2247 | -2.8577 |
| 0.1 | -0.1059 | -1.0991 | 0.9412 | -2.8158 |
| 0.2 | -0.2188 | -1.1602 | 0.6624 | -2.7470 |
| 0.3 | -0.3363 | -1.1656 | 0.3954 | -2.5722 |
| 0.4 | -0.4427 | -0.8993 | 0.1509 | -2.3209 |
| 0.5 | -0.5056 | -0.3395 | -0.0752 | -2.2574 |
| 0.6 | -0.5161 | 0.0728 | -0.3100 | -2.4627 |
| 0.7 | -0.5021 | 0.1645 | -0.5676 | -2.6737 |
| 0.8 | -0.4881 | 0.1060 | -0.8414 | -2.7891 |
| 0.9 | -0.4817 | 0.0240 | -1.1232 | -2.8381 |
| 1.0 | -0.4813 | 0.0025 | -1.4076 | -2.8407 |

Table 69. Converged control variables, SCGRA, Example 4.8.

| t | u | v |
|---|---|---|
| 0.0 | -2.8253 | 0.6091 |
| 0.1 | -1.7058 | 0.3974 |
| 0.2 | -1.7383 | 1.1209 |
| 0.3 | -0.2742 | 2.3718 |
| 0.4 | 3.6856 | 2.1252 |
| 0.5 | 5.3103 | -1.0509 |
| 0.6 | 2.4107 | -2.4698 |
| 0.7 | 0.0279 | -1.6193 |
| 0.8 | -0.7290 | -0.7586 |
| 0.9 | -0.6663 | -0.2584 |
| 1.0 | 0.5504 | 0.2443 |

$\tau = 1.0000$

Table 70. Convergence history, SOGRA, Example 4.8.

| N | Phase | P | Q | I |
|---|---|---|---|---|
| 0 | | 0.28 E+02 | | |
| 1 | REST | 0.87 E+01 | | |
| 2 | REST | 0.59 E-01 | | |
| 3 | REST | 0.19 E-05 | | |
| 4 | REST | 0.30 E-15 | 0.85 E-01 | 0.97910 |
| 5 | GRAD | 0.17 E-02 | | |
| 6 | REST | 0.19 E-08 | 0.32 E-02 | 0.79284 |
| 7 | GRAD | 0.11 E-02 | | |
| 8 | REST | 0.18 E-09 | 0.11 E-02 | 0.75471 |
| 9 | GRAD | 0.12 E-05 | | |
| 10 | REST | 0.13 E-14 | 0.36 E-03 | 0.75132 |
| 11 | GRAD | 0.34 E-05 | | |
| 12 | REST | 0.24 E-14 | 0.57 E-03 | 0.74892 |
| 13 | GRAD | 0.44 E-06 | | |
| 14 | REST | 0.17 E-15 | 0.19 E-03 | 0.74718 |
| 15 | GRAD | 0.10 E-05 | | |
| 16 | REST | 0.13 E-15 | 0.34 E-03 | 0.74585 |
| 17 | GRAD | 0.13 E-06 | | |
| 18 | REST | 0.28 E-16 | 0.11 E-03 | 0.74482 |
| 19 | GRAD | 0.42 E-06 | | |
| 20 | REST | 0.21 E-16 | 0.22 E-03 | 0.74399 |
| 21 | GRAD | 0.87 E-07 | | |
| 22 | REST | 0.55 E-17 | 0.77 E-04 | 0.74332 |

Table 71. Converged state variables, SOGRA, Example 4.8.

| t | x | y | z | h |
|---|---|---|---|---|
| 0.0 | 0.0000 | -1.0000 | 1.2247 | -2.8577 |
| 0.1 | -0.1085 | -1.1435 | 0.9426 | -2.7882 |
| 0.2 | -0.2256 | -1.1872 | 0.6675 | -2.7060 |
| 0.3 | -0.3430 | -1.1382 | 0.4038 | -2.5527 |
| 0.4 | -0.4453 | -0.8539 | 0.1591 | -2.3435 |
| 0.5 | -0.5049 | -0.3254 | -0.0706 | -2.3029 |
| 0.6 | -0.5159 | 0.0519 | -0.3098 | -2.4983 |
| 0.7 | -0.5044 | 0.1432 | -0.5696 | -2.6831 |
| 0.8 | -0.4914 | 0.1087 | -0.8434 | -2.7809 |
| 0.9 | -0.4832 | 0.0585 | -1.1239 | -2.8211 |
| 1.0 | -0.4778 | 0.0670 | -1.4063 | -2.8204 |

Table 72. Converged control variables, SOGRA, Example 4.8.

| t | u | v |
|---|---|---|
| 0.0 | -3.4340 | 0.8575 |
| 0.1 | -1.9189 | 0.6509 |
| 0.2 | -1.2952 | 1.0958 |
| 0.3 | 0.2238 | 1.9870 |
| 0.4 | 3.6035 | 1.7528 |
| 0.5 | 4.9091 | -1.1207 |
| 0.6 | 2.1865 | -2.2304 |
| 0.7 | 0.1680 | -1.3840 |
| 0.8 | -0.4350 | -0.6379 |
| 0.9 | -0.2957 | -0.1943 |
| 1.0 | 0.8164 | 0.2342 |

$\tau = 1.0000$

AD-A051 103 RICE UNIV HOUSTON TEX AERO-ASTRONAUTICS GROUP F/G 12/1
SEQUENTIAL CONJUGATE GRADIENT-RESTORATION ALGORITHM FOR OPTIMAL--ETC(U)
1977 J R CLOUTIER, B P MOHANTY, A MIELE AFOSR-76-3075
UNCLASSIFIED AAR-127 AFOSR-TR-78-0327 NL

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Example 4.9. Consider the following problem with bounded time derivative of the state and free final time:

$$I = \tau, \tag{184}$$

$$\dot{x} = \tau z \cos u, \qquad \dot{y} = \tau z \sin u, \qquad \dot{z} = \tau \sin u, \tag{185}$$

$$1/3 - \dot{y}/\tau \geq 0, \tag{186}$$

$$x(0) = 0, \qquad y(0) = 0, \qquad z(0) = 0, \tag{187}$$

$$x(1) = 1. \tag{188}$$

By means of the transformation techniques of Ref. 9, the previous problem can be converted into the following equality constrained problem:

$$I = \tau, \tag{189}$$

$$\dot{x} = \tau z \cos u, \qquad \dot{y} = \tau z \sin u, \qquad \dot{z} = \tau \sin u, \tag{190}$$

$$1/3 - z \sin u - v^2 = 0, \tag{191}$$

$$x(0) = 0, \qquad y(0) = 0, \qquad z(0) = 0, \tag{192}$$

$$x(1) = 1. \tag{193}$$

Assume the nominal functions

$$x(t) = t, \qquad y(t) = 0, \qquad z(t) = t, \tag{194}$$

$$u(t) = 1, \qquad v(t) = 1, \tag{195}$$

$$\tau = 1. \tag{196}$$

The results for the sequential conjugate gradient-restoration algorithm(SCGRA) of Ref. 1 are given in Tables 73-75. The results for the sequential ordinary gradient-restoration algorithm (SOGRA) of Ref. 2 are given in Tables 76-78. Note that SCGRA leads to the solution in $N = 17$ iterations, while SOGRA leads to the solution in $N = 19$ iterations.

Table 73. Convergence history, SCGRA, Example 4.9.

| N | Phase | $\gamma$ | P | Q | I |
|---|---|---|---|---|---|
| 0 | | | 0.20 E+01 | | |
| 1 | REST | | 0.17 E+01 | | |
| 2 | REST | | 0.71 E-02 | | |
| 3 | REST | | 0.13 E-04 | | |
| 4 | REST | | 0.44 E-08 | 0.32 E+00 | 1.92960 |
| 5 | GRAD | $\gamma = 0$ | 0.22 E-01 | | |
| 6 | REST | | 0.40 E-03 | | |
| 7 | REST | | 0.54 E-07 | | |
| 8 | REST | | 0.65 E-15 | 0.14 E-01 | 1.82694 |
| 9 | GRAD | $\gamma \neq 0$ | 0.25 E-03 | | |
| 10 | REST | | 0.10 E-06 | | |
| 11 | REST | | 0.63 E-15 | 0.21 E-02 | 1.82133 |
| 12 | GRAD | $\gamma \neq 0$ | 0.17 E-04 | | |
| 13 | REST | | 0.89 E-09 | 0.52 E-03 | 1.82053 |
| 14 | GRAD | $\gamma = 0$ | 0.52 E-07 | | |
| 15 | REST | | 0.30 E-12 | 0.25 E-03 | 1.82042 |
| 16 | GRAD | $\gamma \neq 0$ | 0.15 E-05 | | |
| 17 | REST | | 0.32 E-10 | 0.58 E-04 | 1.82027 |

Table 74. Converged state variables, SCGRA, Example 4.9.

| t | x | y | z |
|---|---|---|---|
| 0.0 | 0.0000 | 0.0000 | 0.0000 |
| 0.1 | 0.0019 | 0.0163 | 0.1810 |
| 0.2 | 0.0155 | 0.0634 | 0.3561 |
| 0.3 | 0.0643 | 0.1241 | 0.4982 |
| 0.4 | 0.1449 | 0.1847 | 0.6079 |
| 0.5 | 0.2476 | 0.2454 | 0.7006 |
| 0.6 | 0.3683 | 0.3061 | 0.7824 |
| 0.7 | 0.5047 | 0.3668 | 0.8565 |
| 0.8 | 0.6554 | 0.4267 | 0.9238 |
| 0.9 | 0.8219 | 0.4731 | 0.9727 |
| 1.0 | 1.0000 | 0.4896 | 0.9896 |

Table 75. Converged control variables, SCGRA, Example 4.9.

| t | u | v |
|---|---|---|
| 0.0 | 1.5752 | 0.5773 |
| 0.1 | 1.3931 | 0.3938 |
| 0.2 | 1.2037 | 0.0297 |
| 0.3 | 0.7328 | -0.0038 |
| 0.4 | 0.5803 | 0.0022 |
| 0.5 | 0.4957 | -0.0022 |
| 0.6 | 0.4399 | 0.0088 |
| 0.7 | 0.3996 | -0.0063 |
| 0.8 | 0.3454 | 0.1431 |
| 0.9 | 0.1869 | 0.3906 |
| 1.0 | -0.0011 | 0.5783 |

$\tau = 1.8202$

Table 76. Convergence history, SOGRA, Example 4.9.

| N | Phase | P | Q | I |
|---|---|---|---|---|
| 0 | | 0.20 E+01 | | |
| 1 | REST | 0.17 E+01 | | |
| 2 | REST | 0.71 E-02 | | |
| 3 | REST | 0.13 E-04 | | |
| 4 | REST | 0.44 E-08 | 0.32 E+00 | 1.92960 |
| 5 | GRAD | 0.22 E-01 | | |
| 6 | REST | 0.40 E-03 | | |
| 7 | REST | 0.54 E-07 | | |
| 8 | REST | 0.65 E-15 | 0.14 E-01 | 1.82694 |
| 9 | GRAD | 0.56 E-04 | | |
| 10 | REST | 0.66 E-08 | 0.38 E-02 | 1.82162 |
| 11 | GRAD | 0.13 E-05 | | |
| 12 | REST | 0.18 E-09 | 0.96 E-03 | 1.82084 |
| 13 | GRAD | 0.27 E-06 | | |
| 14 | REST | 0.40 E-11 | 0.60 E-03 | 1.82055 |
| 15 | GRAD | 0.51 E-07 | | |
| 16 | REST | 0.56 E-12 | 0.22 E-03 | 1.82042 |
| 17 | GRAD | 0.23 E-07 | | |
| 18 | REST | 0.47 E-13 | 0.17 E-03 | 1.82035 |
| 19 | GRAD | 0.59 E-08 | 0.77 E-04 | 1.82027 |

Table 77. Converged state variables, SOGRA, Example 4.9.

| t | x | y | z |
|---|---|---|---|
| 0.0 | 0.0000 | 0.0000 | 0.0000 |
| 0.1 | 0.0020 | 0.0163 | 0.1810 |
| 0.2 | 0.0157 | 0.0633 | 0.3559 |
| 0.3 | 0.0644 | 0.1240 | 0.4980 |
| 0.4 | 0.1450 | 0.1847 | 0.6078 |
| 0.5 | 0.2477 | 0.2454 | 0.7005 |
| 0.6 | 0.3684 | 0.3060 | 0.7824 |
| 0.7 | 0.5047 | 0.3667 | 0.8564 |
| 0.8 | 0.6556 | 0.4261 | 0.9232 |
| 0.9 | 0.8222 | 0.4712 | 0.9708 |
| 1.0 | 1.0000 | 0.4874 | 0.9873 |

Table 78. Converged control variables, SOGRA, Example 4.9.

| t | u | v |
|---|---|---|
| 0.0 | 1.5707 | 0.5773 |
| 0.1 | 1.3872 | 0.3941 |
| 0.2 | 1.2035 | 0.0337 |
| 0.3 | 0.7339 | -0.0065 |
| 0.4 | 0.5804 | -0.0004 |
| 0.5 | 0.4958 | 0.0000 |
| 0.6 | 0.4400 | 0.0000 |
| 0.7 | 0.3996 | 0.0109 |
| 0.8 | 0.3382 | 0.1645 |
| 0.9 | 0.1814 | 0.3976 |
| 1.0 | -0.0005 | 0.5778 |

$\tau = 1.8202$

Example 4.10. Consider the following problem with bounded time derivative of the state and free final time:

$$I = \tau, \tag{197}$$

$$\dot{x} = \tau u, \qquad \dot{y} = \tau(u^2 - x^2 - 1/2), \tag{198}$$

$$\dot{y}/\tau + 1/2 \geq 0, \tag{199}$$

$$x(0) = 0, \qquad y(0) = 0, \tag{200}$$

$$x(1) = 1, \qquad y(1) = -\pi/4. \tag{201}$$

By means of the transformation techniques of Ref. 9 , the previous problem can be converted into the following equality constrained problem:

$$I = \tau, \tag{202}$$

$$\dot{x} = \tau u, \qquad \dot{y} = \tau(u^2 - x^2 - 1/2), \tag{203}$$

$$u^2 - x^2 - v^2 = 0, \tag{204}$$

$$x(0) = 0, \qquad y(0) = 0, \tag{205}$$

$$x(1) = 1, \qquad y(1) = -\pi/4. \tag{206}$$

Assume the nominal functions

$$x(t) = t, \qquad y(t) = -(\pi/4)t, \tag{207}$$

$$u(t)=1, \quad v(t)=1, \tag{208}$$

$$\tau=1. \tag{209}$$

The results for the sequential conjugate gradient-restoration algorithm (SCGRA) of Ref. 1 are given in Tables 79-81. The results for the sequential ordinary gradient-restoration algorithm (SOGRA) of Ref. 2 are given in Tables 82-84. Note that SCGRA leads to the solution in $N=12$ iterations, while SOGRA leads to the solution in $N=14$ iterations.

Table 79. Convergence history, SCGRA, Example 4.10.

| N | Phase | $\gamma$ | P | Q | I |
|---|---|---|---|---|---|
| 0 | | | 0.11 E+01 | | |
| 1 | REST | | 0.19 E+00 | | |
| 2 | REST | | 0.11 E+00 | | |
| 3 | REST | | 0.29 E-03 | | |
| 4 | REST | | 0.77 E-07 | | |
| 5 | REST | | 0.22 E-14 | 0.21 E-01 | 1.82848 |
| 6 | GRAD | $\gamma = 0$ | 0.51 E-04 | | |
| 7 | REST | | 0.30 E-07 | | |
| 8 | REST | | 0.47 E-15 | 0.20 E-02 | 1.82290 |
| 9 | GRAD | $\gamma \neq 0$ | 0.32 E-05 | | |
| 10 | REST | | 0.14 E-09 | 0.39 E-03 | 1.82238 |
| 11 | GRAD | $\gamma \neq 0$ | 0.49 E-06 | | |
| 12 | REST | | 0.69 E-11 | 0.97 E-04 | 1.82228 |

Table 80. Converged state variables, SCGRA, Example 4.10.

| t | x | y |
|---|---|---|
| 0.0 | 0.0000 | 0.0000 |
| 0.1 | 0.0906 | -0.0464 |
| 0.2 | 0.1784 | -0.0987 |
| 0.3 | 0.2601 | -0.1621 |
| 0.4 | 0.3330 | -0.2402 |
| 0.5 | 0.4020 | -0.3298 |
| 0.6 | 0.4824 | -0.4209 |
| 0.7 | 0.5788 | -0.5120 |
| 0.8 | 0.6945 | -0.6031 |
| 0.9 | 0.8334 | -0.6942 |
| 1.0 | 1.0000 | -0.7853 |

Table 81. Converged control variables, SCGRA, Example 4.10.

| t | u | v |
|---|---|---|
| 0.0 | 0.5004 | 0.5004 |
| 0.1 | 0.4922 | 0.4838 |
| 0.2 | 0.4677 | 0.4324 |
| 0.3 | 0.4265 | 0.3379 |
| 0.4 | 0.3751 | 0.1725 |
| 0.5 | 0.4020 | 0.0035 |
| 0.6 | 0.4824 | 0.0008 |
| 0.7 | 0.5788 | 0.0044 |
| 0.8 | 0.6945 | 0.0000 |
| 0.9 | 0.8334 | 0.0039 |
| 1.0 | 1.0000 | -0.0087 |

$\tau = 1.8222$

Table 82. Convergence history, SOGRA, Example 4.10.

| N | Phase | P | Q | I |
|---|---|---|---|---|
| 0 | | 0.11 E+01 | | |
| 1 | REST | 0.19 E+00 | | |
| 2 | REST | 0.11 E+00 | | |
| 3 | REST | 0.29 E-03 | | |
| 4 | REST | 0.77 E-07 | | |
| 5 | REST | 0.22 E-14 | 0.21 E-01 | 1.82848 |
| 6 | GRAD | 0.51 E-04 | | |
| 7 | REST | 0.30 E-07 | | |
| 8 | REST | 0.47 E-15 | 0.20 E-02 | 1.82290 |
| 9 | GRAD | 0.28 E-06 | | |
| 10 | REST | 0.83 E-12 | 0.55 E-03 | 1.82245 |
| 11 | GRAD | 0.29 E-07 | | |
| 12 | REST | 0.18 E-13 | 0.22 E-03 | 1.82234 |
| 13 | GRAD | 0.60 E-08 | 0.10 E-03 | 1.82224 |
| 14 | GRAD | 0.77 E-08 | 0.39 E-04 | 1.82222 |

Table 83. Converged state variables, SOGRA, Example 4.10.

| t | x | y |
|---|---|---|
| 0.0 | 0.0000 | 0.0000 |
| 0.1 | 0.0905 | -0.0465 |
| 0.2 | 0.1781 | -0.0989 |
| 0.3 | 0.2598 | -0.1623 |
| 0.4 | 0.3331 | -0.2401 |
| 0.5 | 0.4020 | -0.3298 |
| 0.6 | 0.4824 | -0.4209 |
| 0.7 | 0.5788 | -0.5120 |
| 0.8 | 0.6945 | -0.6031 |
| 0.9 | 0.8334 | -0.6942 |
| 1.0 | 1.0000 | -0.7853 |

Table 84. Converged control variables, SOGRA, Example 4.10.

| t | u | v |
|---|---|---|
| 0.0 | 0.4999 | 0.4999 |
| 0.1 | 0.4916 | 0.4832 |
| 0.2 | 0.4670 | 0.4317 |
| 0.3 | 0.4271 | 0.3389 |
| 0.4 | 0.3768 | 0.1762 |
| 0.5 | 0.4020 | 0.0092 |
| 0.6 | 0.4324 | 0.0000 |
| 0.7 | 0.5788 | 0.0001 |
| 0.8 | 0.6945 | 0.0000 |
| 0.9 | 0.8333 | -0.0007 |
| 1.0 | 0.9996 | 0.0008 |

$\tau = 1.8222$

Example 4.11. Consider the following problem:[8]

$$I = \int_0^1 (x^2 + y^2 + u^2/400 + v^2/400)dt, \tag{210}$$

$$\dot{x} = y, \qquad \dot{y} = (u + v)/2 - y, \tag{211}$$

$$u - v = 0, \tag{212}$$

$$x(0) = 0, \qquad y(0) = -1, \tag{213}$$

$$x^2(1) + y^2(1) - 1 = 0. \tag{214}$$

Assume the nominal functions

$$x(t) = 0, \qquad y(t) = -1, \tag{215}$$

$$u(t) = 0, \qquad v(t) = 0. \tag{216}$$

The results for the sequential conjugate gradient-restoration algorithm (SCGRA) of Ref. 1 are given in Tables 85-87. The results for the sequential ordinary gradient-restoration algorithm (SOGRA) of Ref. 2 are given in Tables 88-90. Note that SCGRA leads to the solution in $N = 25$ iterations, while SOGRA leads to the solution in $N = 23$ iterations.

---

[8] This problem is not of the LQ-type, since the boundary condition ( 214 ) is nonlinear.

Table 85. Convergence history, SCGRA, Example 4.11.

| N | Phase | $\gamma$ | P | Q | I |
|---|---|---|---|---|---|
| 0 | | | 0.20 E+01 | | |
| 1 | REST | | 0.72 E+00 | | |
| 2 | REST | | 0.10 E-01 | | |
| 3 | REST | | 0.69 E-05 | | |
| 4 | REST | | 0.35 E-11 | 0.67 E-01 | 0.75412 |
| 5 | GRAD | $\gamma = 0$ | 0.21 E+01 | | |
| 6 | REST | | 0.75 E-01 | | |
| 7 | REST | | 0.55 E-03 | | |
| 8 | REST | | 0.57 E-07 | | |
| 9 | REST | | 0.67 E-15 | 0.13 E+00 | 0.66451 |
| 10 | GRAD | $\gamma = 0$ | 0.66 E-01 | | |
| 11 | REST | | 0.22 E-03 | | |
| 12 | REST | | 0.40 E-08 | 0.74 E-02 | 0.22157 |
| 13 | GRAD | $\gamma \neq 0$ | 0.24 E-02 | | |
| 14 | REST | | 0.39 E-06 | | |
| 15 | REST | | 0.11 E-13 | 0.10 E-02 | 0.18399 |
| 16 | GRAD | $\gamma \neq 0$ | 0.17 E-02 | | |
| 17 | REST | | 0.19 E-06 | | |
| 18 | REST | | 0.25 E-14 | 0.18 E-02 | 0.16398 |
| 19 | GRAD | $\gamma \neq 0$ | 0.20 E-02 | | |
| 20 | REST | | 0.29 E-06 | | |
| 21 | REST | | 0.63 E-14 | 0.25 E-03 | 0.15077 |
| 22 | GRAD | $\gamma \neq 0$ | 0.79 E-04 | | |
| 23 | REST | | 0.50 E-09 | 0.25 E-03 | 0.14896 |
| 24 | GRAD | $\gamma \neq 0$ | 0.12 E-03 | | |
| 25 | REST | | 0.11 E-08 | 0.69 E-04 | 0.14640 |

Table 86. Converged state variables, SCGRA, Example 4.11.

| t | x | y |
|---|---|---|
| 0.0 | 0.0000 | -1.0000 |
| 0.1 | -0.0504 | -0.1709 |
| 0.2 | -0.0542 | 0.0387 |
| 0.3 | -0.0490 | 0.0497 |
| 0.4 | -0.0455 | 0.0190 |
| 0.5 | -0.0450 | -0.0046 |
| 0.6 | -0.0458 | -0.0089 |
| 0.7 | -0.0462 | 0.0014 |
| 0.8 | -0.0464 | -0.0133 |
| 0.9 | -0.0557 | -0.2213 |
| 1.0 | -0.1092 | -0.9940 |

Table 87. Converged control variables, SCGRA, Example 4.11.

| t | u | v |
|---|---|---|
| 0.0 | 12.9549 | 12.9549 |
| 0.1 | 3.9884 | 3.9884 |
| 0.2 | 0.7298 | 0.7298 |
| 0.3 | -0.1849 | -0.1849 |
| 0.4 | -0.2896 | -0.2896 |
| 0.5 | -0.1524 | -0.1524 |
| 0.6 | 0.0486 | 0.0486 |
| 0.7 | 0.0886 | 0.0886 |
| 0.8 | -0.7372 | -0.7372 |
| 0.9 | -4.1725 | -4.1725 |
| 1.0 | -13.8279 | -13.8279 |

$\tau = 1.0000$

Table 88. Convergence history, SOGRA, Example 4.11.

| N | Phase | P | Q | I |
|---|---|---|---|---|
| 0 | | 0.20 E+01 | | |
| 1 | REST | 0.72 E+00 | | |
| 2 | REST | 0.10 E-01 | | |
| 3 | REST | 0.69 E-05 | | |
| 4 | REST | 0.35 E-11 | 0.67 E-01 | 0.75412 |
| 5 | GRAD | 0.21 E+01 | | |
| 6 | REST | 0.75 E-01 | | |
| 7 | REST | 0.55 E-03 | | |
| 8 | REST | 0.57 E-07 | | |
| 9 | REST | 0.67 E-15 | 0.13 E+00 | 0.66451 |
| 10 | GRAD | 0.66 E-01 | | |
| 11 | REST | 0.22 E-03 | | |
| 12 | REST | 0.40 E-08 | 0.74 E-02 | 0.22157 |
| 13 | GRAD | 0.39 E-03 | | |
| 14 | REST | 0.10 E-07 | | |
| 15 | REST | 0.84 E-17 | 0.10 E-02 | 0.18150 |
| 16 | GRAD | 0.15 E-03 | | |
| 17 | REST | 0.19 E-08 | 0.15 E-02 | 0.15905 |
| 18 | GRAD | 0.17 E-04 | | |
| 19 | REST | 0.22 E-10 | 0.13 E-03 | 0.15158 |
| 20 | GRAD | 0.28 E-05 | | |
| 21 | REST | 0.65 E-12 | 0.26 E-03 | 0.14756 |
| 22 | GRAD | 0.53 E-06 | | |
| 23 | REST | 0.22 E-13 | 0.24 E-04 | 0.14627 |

Table 89. Converged state variables, SOGRA, Example 4.11.

| t | x | y |
|---|---|---|
| 0.0 | 0.0000 | -1.0000 |
| 0.1 | -0.0546 | -0.2306 |
| 0.2 | -0.0637 | -0.0041 |
| 0.3 | -0.0615 | 0.0324 |
| 0.4 | -0.0588 | 0.0199 |
| 0.5 | -0.0575 | 0.0062 |
| 0.6 | -0.0572 | 0.0027 |
| 0.7 | -0.0569 | -0.0002 |
| 0.8 | -0.0586 | -0.0490 |
| 0.9 | -0.0724 | -0.2750 |
| 1.0 | -0.1296 | -0.9915 |

Table 90. Converged control variables, SOGRA, Example 4.11.

| t | u | v |
|---|---|---|
| 0.0 | 11.5129 | 11.5129 |
| 0.1 | 3.8943 | 3.8943 |
| 0.2 | 0.9527 | 0.9527 |
| 0.3 | 0.0225 | 0.0225 |
| 0.4 | -0.1490 | -0.1490 |
| 0.5 | -0.0808 | -0.0808 |
| 0.6 | 0.0061 | 0.0061 |
| 0.7 | -0.1338 | -0.1338 |
| 0.8 | -1.0940 | -1.0940 |
| 0.9 | -4.2466 | -4.2466 |
| 1.0 | -12.4066 | -12.4066 |

$\tau = 1.0000$

Example 4.12. Consider the following problem:[9]

$$I = \int_0^1 (x^2 + y^2 + u^2/200 + v^2)\,dt, \tag{217}$$

$$\dot{x} = y, \qquad \dot{y} = u + v - y, \tag{218}$$

$$u + 2v - 10/(1 + 10t)^2 = 0, \tag{219}$$

$$x(0) = 0, \qquad y(0) = -1, \tag{220}$$

$$x^4(1) + y^4(1) - 1 = 0. \tag{221}$$

Assume the nominal functions

$$x(t) = 0, \qquad y(t) = -1, \tag{222}$$

$$u(t) = 0, \qquad v(t) = 0. \tag{223}$$

The results for the sequential conjugate gradient-restoration algorithm(SCGRA) of Ref. 1 are given in Tables 91-93. The results for the sequential ordinary gradient-restoration algorithm (SOGRA) of Ref. 2 are given in Tables 94-96. Note that both SCGRA and SOGRA lead to the solution in $N = 9$ iterations.

[9]This problem is not of the LQ-type, since the boundary condition (221) is nonlinear.

Table 91. Convergence history, SCGRA, Example 4.12.

| N | Phase | $\gamma$ | P | Q | I |
|---|---|---|---|---|---|
| 0 | | | 0.53 E+01 | | |
| 1 | REST | | 0.82 E-01 | | |
| 2 | REST | | 0.55 E-03 | | |
| 3 | REST | | 0.42 E-07 | | |
| 4 | REST | | 0.26 E-15 | 0.35 E+00 | 3.30976 |
| 5 | GRAD | $\gamma = 0$ | 0.14 E-02 | | |
| 6 | REST | | 0.26 E-06 | | |
| 7 | REST | | 0.97 E-14 | 0.25 E-02 | 2.84730 |
| 8 | GRAD | $\gamma \neq 0$ | 0.58 E-06 | | |
| 9 | REST | | 0.48 E-13 | 0.24 E-05 | 2.84404 |

Table 92. Converged state variables, SCGRA, Example 4.12.

| t | x | y |
|---|---|---|
| 0.0 | 0.0000 | -1.0000 |
| 0.1 | -0.0697 | -0.5171 |
| 0.2 | -0.1143 | -0.4013 |
| 0.3 | -0.1531 | -0.3850 |
| 0.4 | -0.1928 | -0.4127 |
| 0.5 | -0.2365 | -0.4662 |
| 0.6 | -0.2866 | -0.5378 |
| 0.7 | -0.3446 | -0.6245 |
| 0.8 | -0.4120 | -0.7253 |
| 0.9 | -0.4902 | -0.8403 |
| 1.0 | -0.5806 | -0.9702 |

Table 93. Converged control variables, SCGRA, Example 4.12.

| t | u | v |
|---|---|---|
| 0.0 | 8.3084 | 0.8457 |
| 0.1 | 0.6854 | 0.9072 |
| 0.2 | -0.8977 | 1.0044 |
| 0.3 | -1.6023 | 1.1136 |
| 0.4 | -2.0737 | 1.2368 |
| 0.5 | -2.4761 | 1.3769 |
| 0.6 | -2.8695 | 1.5367 |
| 0.7 | -3.2820 | 1.7191 |
| 0.8 | -3.7303 | 1.9269 |
| 0.9 | -4.2261 | 2.1630 |
| 1.0 | -4.7792 | 2.4309 |

$\tau = 1.0000$

Table 94. Convergence history, SOGRA, Example 4.12.

| N | Phase | P | Ω | I |
|---|---|---|---|---|
| 0 | | 0.53 E+01 | | |
| 1 | REST | 0.82 E-01 | | |
| 2 | REST | 0.55 E-03 | | |
| 3 | REST | 0.42 E-07 | | |
| 4 | REST | 0.26 E-15 | 0.35 E+00 | 3.30976 |
| 5 | GRAD | 0.14 E-02 | | |
| 6 | REST | 0.26 E-06 | | |
| 7 | REST | 0.97 E-14 | 0.25 E-02 | 2.84730 |
| 8 | GRAD | 0.49 E-07 | | |
| 9 | REST | 0.34 E-15 | 0.87 E-05 | 2.84404 |

Table 95. Converged state variables, SOGRA, Example 4.12.

| t | x | y |
|---|---|---|
| 0.0 | 0.0000 | -1.0000 |
| 0.1 | -0.0698 | -0.5183 |
| 0.2 | -0.1145 | -0.4028 |
| 0.3 | -0.1535 | -0.3864 |
| 0.4 | -0.1933 | -0.4141 |
| 0.5 | -0.2372 | -0.4674 |
| 0.6 | -0.2873 | -0.5388 |
| 0.7 | -0.3454 | -0.6253 |
| 0.8 | -0.4129 | -0.7258 |
| 0.9 | -0.4911 | -0.8405 |
| 1.0 | -0.5815 | -0.9700 |

Table 96. Converged control variables, SOGRA, Example 4.12.

| t | u | v |
|---|---|---|
| 0.0 | 8.2562 | 0.8718 |
| 0.1 | 0.6732 | 0.9133 |
| 0.2 | -0.9023 | 1.0067 |
| 0.3 | -1.6040 | 1.1145 |
| 0.4 | -2.0738 | 1.2369 |
| 0.5 | -2.4749 | 1.3763 |
| 0.6 | -2.8671 | 1.5356 |
| 0.7 | -3.2785 | 1.7173 |
| 0.8 | -3.7254 | 1.9244 |
| 0.9 | -4.2196 | 2.1598 |
| 1.0 | -4.7708 | 2.4267 |

$\tau = 1.0000$

## 5. Discussion and Conclusions

In Ref. 1, Cloutier, Mohanty, and Miele developed the sequential conjugate gradient-restoration algorithm for minimizing a functional subject to differential constraints, nondifferential constraints, and terminal constraints. In this report, 16 numerical examples are presented, 4 pertaining to a quadratic functional subject to linear constraints, and 12 pertaining to a nonquadratic functional subject to nonlinear constraints. These examples demonstrate the feasibility as well as the rapid convergence characteristics of the sequential conjugate gradient-restoration algorithm.

Comparative results concerning the sequential conjugate gradient-restoration algorithm (SCGRA) and the sequential ordinary gradient-restoration algorithm (SOGRA) are presented in Tables 97-106, where the number of iterations required to achieve different levels of the error in the optimality conditions Q is given for a tolerance level of the constraint error $P \leq E\text{-}08$. More precisely, results are presented for the following tolerance levels:

$$P \leq E\text{-}08\,, \qquad Q \leq E\text{-}02\,, \qquad \text{Tables 97-98}\,; \tag{224}$$

$$P \leq E\text{-}08\,, \qquad Q \leq E\text{-}03\,, \qquad \text{Tables 99-100}\,; \tag{225}$$

$$P \leq E\text{-}08\,, \qquad Q \leq E\text{-}04\,, \qquad \text{Tables 101-102}\,; \tag{226}$$

$$P \leq 08\,, \qquad Q \leq E\text{-}05\,, \qquad \text{Tables 103-104}\,; \qquad (227)$$

$$P \leq 08\,, \qquad Q \leq E\text{-}06\,, \qquad \text{Tables 105-106}\,. \qquad (228)$$

Cumulative results for the 4 linear-quadratic problems and the 12 nonlinear-nonquadratic problems investigated are given in Tables 107-108: here, the total number of iterations for convergence $\Sigma N$ is presented as a function of the tolerance level chosen for the error in the optimality conditions Q. These cumulative results are presented for only those tolerance levels such that convergence occurs with both algorithms in all of the examples being investigated.

Linear-Quadratic Problems. For these problems, Table 107 shows the clear-cut superiority of SCGRA with respect to SOGRA. The saving in number of iterations is negligible if the tolerance level is rather relaxed ($Q \leq E\text{-}03$), is 29% for $Q \leq E\text{-}04$, is 53% for $Q \leq E\text{-}05$, and is 60% for $Q \leq E\text{-}06$.

Nonlinear-Nonquadratic Problems. For these problems, Table 108 shows that the superiority of SCGRA with respect to SOGRA is not clear-cut. There is no saving in number of iterations for $Q \leq E\text{-}03$, and the saving is 12% for $Q \leq E\text{-}04$. Data for $Q \leq E\text{-}05$ and $Q \leq E\text{-}06$ are not given, because both algorithms fail to attain these tolerance levels in some of the test problems.

Remarks. With reference to the previous discussion,

the following remarks are pertinent.

(i) The experiments performed show that the computer time per iteration is roughly the same for SCGRA and SOGRA. Therefore, the conclusions pertaining to savings in number of iterations also apply to savings in computer time.

(ii) Generally speaking, the relative advantage of SCGRA with respect to SOGRA depends strongly on the problem being solved. In particular, it is greater for problems with free endpoints than for problems with fixed endpoints.

(iii) For linear-quadratic problems of optimal control, there seems to be a clear advantage in the use of conjugate gradient techniques. On the other hand, for nonlinear-nonquadratic problems, the advantage of conjugate gradient techniques is not as clear-cut as for linear-quadratic problems.

The probable explanation for this behaviour is as follows. Essentially, an optimal control problem can be viewed to be equivalent to a mathematical programming problem with a large number of degrees of freedom. In turn, the number of degrees of freedom is of the same order as the product of the number of integration steps times the number of independent controls. For a problem where the number of integration steps is 50 and the number of independent controls is one, the number of degrees of freedom is of order 50. This means

that, for the conjugate gradient properties to become manifest, the number of iterations would have to be some multiple of 50, which is a prohibitive number for problems of optimal control. Usually, both ordinary gradient techniques and conjugate gradient techniques enable one to achieve convergence in less than 50 iterations if the tolerance level set for the error in the optimality conditions is not excessively small.

Table 97. Number of iterations to $P \leq E-08$ and $Q \leq E-02$, linear-quadratic problems.

| Example | SCGRA | SOGRA |
|---|---|---|
| 3.1 | 2 | 2 |
| 3.2 | 2 | 2 |
| 3.3 | 1 | 1 |
| 3.4 | 1 | 1 |

Table 98. Number of iterations to $P \leq E-08$ and $Q \leq E-02$, nonlinear-nonquadratic problems.

| Example | SCGRA | SOGRA |
|---|---|---|
| 4.1 | 8 | 8 |
| 4.2 | 10 | 10 |
| 4.3 | 7 | 7 |
| 4.4 | 5 | 5 |
| 4.5 | 7 | 7 |
| 4.6 | 8 | 8 |
| 4.7 | 10 | 10 |
| 4.8 | 6 | 6 |
| 4.9 | 11 | 10 |
| 4.10 | 8 | 8 |
| 4.11 | 12 | 12 |
| 4.12 | 7 | 7 |

Table 99. Number of iterations to $P \leq E\text{-}08$ and $Q \leq E\text{-}03$, linear quadratic problems.

| Example | SCGRA | SOGRA |
|---|---|---|
| 3.1 | 3 | 4 |
| 3.2 | 3 | 3 |
| 3.3 | 2 | 2 |
| 3.4 | 2 | 2 |

Table 100. Number of iterations to $P \leq E\text{-}08$ and $Q \leq E\text{-}03$, nonlinear-nonquadratic problems.

| Example | SCGRA | SOGRA |
|---|---|---|
| 4.1 | 8 | 8 |
| 4.2 | 16 | 18 |
| 4.3 | 13 | 13 |
| 4.4 | 9 | 11 |
| 4.5 | 11 | 11 |
| 4.6 | 10 | 14 |
| 4.7 | 19 | 15 |
| 4.8 | 10 | 10 |
| 4.9 | 13 | 12 |
| 4.10 | 10 | 10 |
| 4.11 | 21 | 19 |
| 4.12 | 9 | 9 |

Table 101. Number of iterations to $P \leq$ E-08 and $Q \leq$ E-04, linear-quadratic problems.

| Example | SCGRA | SOGRA |
|---|---|---|
| 3.1 | 4 | 3 |
| 3.2 | 3 | 3 |
| 3.3 | 3 | 4 |
| 3.4 | 2 | 2 |

Table 102. Number of iterations to $P \leq$ E-08 and $Q \leq$ E-04, nonlinear-nonquadratic problems.

| Example | SCGRA | SOGRA |
|---|---|---|
| 4.1 | 14 | 12 |
| 4.2 | 28 | 34 |
| 4.3 | 17 | 24 |
| 4.4 | 16 | 21 |
| 4.5 | 15 | 16 |
| 4.6 | 20 | 24 |
| 4.7 | 23 | 19 |
| 4.8 | 12 | 22 |
| 4.9 | 17 | 19 |
| 4.10 | 12 | 14 |
| 4.11 | 25 | 23 |
| 4.12 | 9 | 9 |

Table 103. Number of iterations to $P \leq$ E-08 and $Q \leq$ E-05, linear-quadratic problems.

| Example | SCGRA | SOGRA |
|---|---|---|
| 3.1 | 5 | 16 |
| 3.2 | 3 | 4 |
| 3.3 | 4 | 8 |
| 3.4 | 2 | 2 |

Table 104. Number of iterations to $P \leq$ E-08 and $Q \leq$ E-05, nonlinear-nonquadratic problems.

| Example | SCGRA | SOGRA |
|---|---|---|
| 4.1 | 22 | 23 |
| 4.2 | 40 | 46 |
| 4.3 | 33 | 29 |
| 4.4 | (*) | 30 |
| 4.5 | 24 | 21 |
| 4.6 | (*) | 27 |
| 4.7 | 31 | 27 |
| 4.8 | 22 | 51 |
| 4.9 | 22 | 23 |
| 4.10 | 18 | (*) |
| 4.11 | 31 | 27 |
| 4.12 | 9 | 9 |

(*) Algorithm unable to achieve given stopping condition.

Table 105. Number of iterations to $P \leq E-08$ and $Q \leq E-06$, linear-quadratic problems.

| Example | SCGRA | SOGRA |
|---|---|---|
| 3.1 | 6 | 26 |
| 3.2 | 4 | 4 |
| 3.3 | 5 | 10 |
| 3.4 | 2 | 2 |

Table 106. Number of iterations to $P \leq E-08$ and $Q \leq E-06$, nonlinear-nonquadratic problems.

| Example | SCGRA | SOGRA |
|---|---|---|
| 4.1 | 30 | 25 |
| 4.2 | 56 | 60 |
| 4.3 | 43 | 53 |
| 4.4 | (*) | (*) |
| 4.5 | (*) | 29 |
| 4.6 | (*) | (*) |
| 4.7 | 45 | 31 |
| 4.8 | 48 | (*) |
| 4.9 | (*) | 29 |
| 4.10 | 20 | (*) |
| 4.11 | 35 | 34 |
| 4.12 | 10 | 10 |

(*) Algorithm unable to achieve given stopping condition.

Table 107. Cumulative number of iterations for convergence $\Sigma N$, linear-quadratic problems.

| Q | SCGRA | SOGRA |
|---|---|---|
| $Q \leq$ E-02 | 6 | 6 |
| $Q \leq$ E-03 | 10 | 11 |
| $Q \leq$ E-04 | 12 | 17 |
| $Q \leq$ E-05 | 14 | 30 |
| $Q \leq$ E-06 | 17 | 42 |

Table 108. Cumulative number of iterations for convergence $\Sigma N$, nonlinear-nonquadratic problems.

| Q | SCGRA | SOGRA |
|---|---|---|
| $Q \leq$ E-02 | 99 | 98 |
| $Q \leq$ E-03 | 149 | 150 |
| $Q \leq$ E-04 | 208 | 237 |

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